Lecture Synopsis 3-14-17

Intermediate Macro Theory

March 21, 2017

1 Production

In the long run, output depends primarily on the resources the economy has to put toward production, namely, Capital (K), Labor (L) and Technology (A). We denote output as Y. To represent the relationship, we can write a **general production function**:

$$Y = F(N, K, A)$$

This is just saying that output is a function of those 3 things, but doesn't specify the specific relationship. If we want to specify one, we can have a **specific production function**:

$$Y = A * N^{1/2} * K * 3/4$$

A general function just captures the idea that output comes from capital, labor, and technology. A specific function takes information about capital, labor, and technology, and tells you a level of output.

1.1 Marginal Product

However, we're interested in more than just the level of output- we'd like to know how it would change if we added a little labor, or a little capital, or took some away. The change in output as a result of a change in capital, holding everything else constant, is the **Marginal Product of Capital**:

$$MPK = \frac{dY}{dK}$$

Similarly, if we add some labor,holding everything else constant, the change in output that results is given by :

$$MPL = \frac{dY}{dN}$$

We generally restrict both marginal product functions to be strictly positive, but diminishing. This essentially just says that as we add more workers, we can always find more stuff for them to do, but those tasks may be less and less valuable in terms of their contribution to output, because we only have some fixed level of capital and technology for them to use. If we add more and more machines, they are always positive in their contribution, but if we don't add more workers, they aren't going to each contribute as much as the first few we add. Mathematically, we represent these restrictions as :

$$\frac{dY}{dK} > 0 \;\; \frac{dY}{dN} > 0$$

$$\frac{d^2Y}{dK^2} < 0 \;\; \frac{d^2Y}{dN^2} < 0$$

A very handy production function is the **Cobb-Douglas** function:

$$Y = N^{\alpha}K^{\beta}, \;\alpha + \beta = 1, \;\alpha > 0, \;\beta > 0$$

We can easily verify that , in general, this function satisfies the derivative conditions above. I will show it for labor, but you should try to do it for capital too as a **challenge.**

$$MPK = \frac{dY}{dK} \tag{1}$$

$$= \frac{d(N^{\alpha}K^{\beta})}{dK} \tag{2}$$

$$= \beta N^{\alpha}K^{\beta-1} \tag{3}$$

Since beta is always positive, and exponents between 0 and 1 always return positive values for a positive base, and the product of 3 positives is positive, we know this is positive. Now, what about the 2nd derivative?

$$\frac{d^2Y}{dK^2} = \frac{d(\beta N^{\alpha}K^{\beta-1})}{dK} \tag{4}$$

$$= \beta(\beta - 1)N^{\alpha}K^{\beta-2} \tag{5}$$

$$= \beta(\beta - 1)N^{\alpha}\frac{K^{\beta}}{K^2} \tag{6}$$

$$\text{signs:}(+)(-)(+)\frac{(+)}{(+)} = (-) \tag{7}$$

$$\rightarrow \frac{d^2Y}{dK^2} < 0 \tag{8}$$

We had to use the **power rule** to take these derivatives. If that's not familiar, or any of the differentiation we did earlier, please check out http://tutorial.math.lamar.edu/Classes/CalcI/DiffFormulas.aspx, Or google "Paul's Online Math Notes Diff Formulas". It's a great resource to refresh on math.

Tip: For the exams, I basically need to see that you

1. Understand Marginal Product is equal to the right derivative

2. Can take that derivative, correctly. You NEED to know your calculus- many people last semester tried to take the exam without a good grasp on differentiation.

3. Can give some kind of justification for the sign you determine- I don't need a crazy proof, but at least put pluses and minuses to indicate the sign of each part, and an explanation.

Challenge: Verify the signs of the first and second derivatives for Labor with Cobb Douglass.

1.2 Returns to Scale

Returns to scale is a description of what happens when we scale up **all** inputs by multiplying them by some factor strictly greater than 1. For example, if we double all inputs, do we double output, less than double output, or more than double output? We have three types of returns to scale:

Increasing Returns to Scale Is when the increase in output from the scaling up inputs by some factor $\lambda > 1$ increases output by more than λ. That is , $F(\lambda N, \lambda K, A) > \lambda F(N, K, A)$.. I get a "more than proportional" increase in output.

Decreasing Returns to Scale Is when the increase in output from the scaling up inputs by some factor $\lambda > 1$ increases output by LESS than λ That is , $F(\lambda N, \lambda K, A) < \lambda F(N, K, A)$.. I get a "less than proportional" increase in output.

Constant Returns to Scale Is when the increase in output from the scaling up inputs by some factor $\lambda > 1$ increases output by EXACTLY λ That is , $F(\lambda N, \lambda K, A) = \lambda F(N, K, A)$.. I get a "perfectly proportional" increase in output.

We can check this property for some particular values (like $\lambda = 2, N = 10, K = 5$) , purely as an example. BUT **no matter how many numbers you plug in, these are examples, not proofs!!!**. To *prove* a property, you need to show one of the above with generic N,K, and λ.

Generally, this is what you need to do. First, establish what it is we want to compare. $Y_1 = \lambda F(N, K, A)$ $Y_2 = F(\lambda N, \lambda K, A)$ Now, we want to simplify Y_2 to get it to be somehow directly comparable to Y_1. This may mean factoring out λ, or distributing some exponents, or whatever else. But you generally need to get Y_2 to be an expression in terms of Y_1, so then we can see how they compare. This takes some practice, but consider the Cobb-Douglas case to start with :

$$Y_1 = N^\alpha K^\beta \tag{9}$$
$$Y_2 = (\lambda N)^\alpha (\lambda K)^\beta \tag{10}$$
$$Y_2 = \lambda^\alpha * N^\alpha * \lambda^\beta * K^\beta \tag{11}$$
$$Y_2 = \lambda^\alpha \lambda^\beta N^\alpha K^\beta \tag{12}$$
$$Y_2 = \lambda^{\alpha+\beta} N^\alpha K^\beta \tag{13}$$
$$Y_2 = \lambda^{\alpha+\beta} * Y_1 \tag{14}$$
$$\text{Since}\alpha + \beta = 1 \tag{15}$$
$$Y_2 = \lambda * Y_1 \tag{16}$$
$$\tag{17}$$

So, we can see clearly that Cobb-Douglass as specified in these notes is always a constant returns to scale function.

Note that we did not need to, nor should we, plug in any values. Do NOT plug in an N, a K, or a λ. The only values you should ever be putting in are parameters, like $\alpha \& \beta$, which are given to you as part of the production function .But it will never be correct to plug in K, N, or λ. Our goal is always to get $Y_2 = F(\lambda N, \lambda K)$ and $Y_1 = \lambda F(N, K)$ in the most general form possible, and compare their size, in accordance with the definitions above.

Challenge : Consider a case where $\alpha + \beta > 1, A = 3$ for the production function $Y = A * N^{1/2} * K * 3/4$. What type of returns to scale does this production function have? What if $A = .5$ instead?