The Stabilizing Effects of Disagreement

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1 Introduction

The spatial interpretation of politics is intuitive and pervasive. Voters frequently talk about a candidate being too far to the right on one issue, or left-leaning on another. Commentators describe how a certain candidate has moved away from his base and toward the center. The spatial interpretation also provides direct guidance for formal theorists on how to model political contests. Positions are points in space, and candidates take those positions. Voters vote for the closest candidate to their ideal points. Since Downs' formalization of Hotelling's initial suggestion that spatial competition could describe two candidate competition, of the one-dimensional spatial model (Downs, 1957; Hotelling, 1929), and the development of the famed Median Voter Theorem by Black (1958), the spatial interpretation of politics has dominated the formal theory of general elections.

The one-dimensional spatial model delivers a powerful prediction: For an odd number of voters, there is a single point, known as a Condorcet Winner, which defeats all others in pairwise voting. This point is the ideal policy of the median voter. Stability comes from the fact that, at the median, half of the remaining voters would like to go left, and half of the remaining voters would like to go right. Since left and right are the only directions to deviate from the Condorcet Winner, there is no way to make more than half of the voters better off.

In higher dimensions, an alternative policy can be in infinitely many directions from the proposal, and at any distance in that direction, including infinitesimal. For any direction of deviation from a proposed Condorcet Winner, it must be the case that at least half of voters will be made worse off by any alternative policy in that direction- even if that alternative is infinitely close to the proposed Condorcet Winner. For each voter who would like to move away from the proposed Condorcet Winner in some direction by any magnitude, there is another voter who would be made worse off, and this has to be true for every direction of deviation in the space. These requirements are characterized by the Plott conditions on the location of voters around that point: they must be radially symmetric, with equal numbers of voters on either side for any possible direction (Plott, 1967).

Two forces drive the Plott conditions: that alternatives can be proposed in any direction, and that alternatives can be arbitrarily close so that gradient criteria are appropriate. A natural conjecture then is to consider the spatial model when the policy space is a convex subset of the preference space, so that candidates cannot promise anything they would like (Matthews, 1980). In the interior of the set, at any potential equilibrium, there can be alternatives arbitrarily close, and in any direction, and so the Plott conditions hold. At boundary points, there is at least a restriction on how many directions matter. Unfortunately, even at boundary points, an equilibrium is unlikely to exist, except when the set of possible alternatives is pathologically pointed. Another approach is to restrict the directions of alternatives that can be compared explicitly (Shepsle and Weingast, 1981). Institutions constrain the directions of deviation by restricting voting to one issue at a time, thereby returning each individual vote to a one dimensional context, meaning that existence of an equilibrium is guaranteed under mild conditions on preferences. However, there is no way to elect a mayor or a president one issue at a time, so many important situations are still potentially doomed to not have an equilibrium. Aside from restricting directions, one can also consider restricting the minimum distance of alternative proposals (Sloss, 1973). Infinitesimal deviations are no longer possible, and so points that are "almost" total medians may sustain a position as an equilibrium. A closely related concept is the Finagle point- a point that either defeats any alternative or is close to another point that defeats the alternative (Wuffle et al., 1989). While each solution is a set of alternatives rather than one, it does imply that even when preferences are not pairwise symmetric, a candidate can likely find a position that is at least readily defensible.

In this paper, we consider a restriction on both directions and distance, the political party of the candidates. Candidates for office rarely run without any affiliation; they part of or sponsored by a party. The party, in turn, restricts the positions that the candidate may take. For example, If a candidate chooses to run for office as a Republican, she may be unable to hold a pro-gun control stance of any kind. It may be that the most moderate position she can take is lukewarmly in favor of gun rights. On defense, she may be unable to promise a budget reduction or even a reduction in the rate of growth- instead, it may be that the most moderate stance she can take is to maintain current increases in spending. A Democrat faces the opposite constraints; he may be unable to advocate for anything less than mandatory background checks and at least marginal spending cuts in defense. The two candidates have some freedom to pick positions in this set- they can choose to be more or less extreme on each issue- but they cannot promise anything in the policy space. The party structure clearly restricts distance, as a Democrat cannot get arbitrarily close to a Republican pledging to double the defense budget and overturn waiting periods for gun purchases. This restriction on the location of the Democrat's position is a restriction on directions as well. The Democrat certainly cannot pledge a larger defense budget increase, or even weaker gun laws. Rather, the Republican only needs to worry about how voters like her own position relative to positions involving less spending, stronger gun laws, or both. Even if it were the case that a majority of voters would like even greater spending increases, there is no one to offer them.

Parties certainly do not exercise total control. Many issues emerge mid-campaign on which the parties do not have an explicit restriction. There times in which candidates go rogue and deviate beyond what is typically allowed by the party. The admissible sets of policies may have significant overlap on certain doorstep issues, like K-12 education or funding for police. The idea that parties generate disjoint sets of alternatives is a polar case to competition with common sets of alternatives, and it allows us to most clearly illustrate a novel aspect of electoral competition. In our analysis, we do not require the candidates to have disjoint sets of alternatives when characterizing our results. Rather, when a proposed equilibrium point is on the boundary of or inside the opponent's set of alternatives, then the situation is the same as that in Matthews (1980) or Plott (1967) respectively, and so the conditions that must hold there are already well understood. The main contribution of this paper is to analyze what happens when one candidate takes a position that the other candidate cannot.

We keep almost all of the typical assumptions of spatial voting in multiple dimensions. We assume a simultaneous game. The candidates are purely office motivated. There is no valence or uncertainty, and voters vote sincerely for the platform closest, in terms of Euclidean Distance, to their ideal points. The only perturbation we are making to the typical multidimensional spatial model is to restrict which alternatives each candidate can propose. Our goal is to identify when a policy for one of the candidates can defeat any of the opposing candidate's policies.

The main results are as follows. For any policy a not in or on the opponent's policy set, we can define a set of "guaranteed supporters" of a. Any voter with an ideal point in the guaranteed supporter set will always prefer a to anything the other candidate can promise. The necessary and sufficient condition for a to defeat any alternative the opponent can promise is that the median voter in each direction is a guaranteed supporter of a. A total median requires the median voter in each direction to be at an identical point while in our model, the median voters only have to be in a region with nonempty interior. Second, if we perturb the distribution of voters, in a direction "favorable" to a, then a still defeats all of the alternatives the opponent can offer. Further, it will be robust to small perturbations in an unfavorable direction as well. Lastly, we derive the guaranteed supporter set for a typical interpretation of US politics and derive some sufficient conditions for existence, which show that an equilibrium is sure to exist whenever voter opinion is not too neutral, and not too far out of line with the existing party structure.

In Section 2, we review the most closely related literature. In Section 3, we introduce the preliminaries of the model and the solution concept we will use. In Section 4, we derive the necessary and sufficient condition for a policy to be an equilibrium outcome. In Section 5, we consider possible perturbations of the distribution of voters. In Section 6, we illustrate an application of the model to a typical interpretation of US politics, and discuss sufficient conditions for existence.

2 Existing Literature

This paper is related to the literature on differentiated candidates in multidimensional spatial competition. Candidates may be differentiated in either a vertical or horizontal way. Vertical differentiation refers to a quality difference between candidates, something that makes all voters prefer one candidate to the other, often referred to as valence. Horizontal differentiation refers to candidates that are merely different- there is no uniform effect of their special traits on all voters. Our model, while similar, assumes a differentiation of what policies can be offered, rather than an extra, exogenous dimension for quality or traits.

Within the vertical differentiation literature, the closest model to the present one is (Ansolabehere and Snyder, 2000). There, one candidate has a quality advantage of some magnitude, and consequently, voters who may otherwise vote against the candidate's position vote for it, in order to get the higher quality politician who is offering it. Even when a total median does not exist, there may still be an equilibrium as long as voter's preferences are not too diverse relative to the size of the valence advantage. Peress (2010) considers the issue of office and policy motivated candidates in multiple dimensions with a valence advantage and vote share maximization, and finds that there is an equilibrium under modest restrictions on the distribution of voters.

In the horizontal differentiation literature, the seminal papers (Krasa and Polborn, 2010, 2014; Dziubiński and Roy, 2011) assume that candidates have taken a fixed position on one or more policy issues. A given spatial position can be an advantage or disadvantage depending on the voter's ideal point. However, since the policy choice in one dimension is fixed, while there are nominally two dimensions, candidates are only choosing in one. An exception to this is Xefteris (2017), where candidates are flexible on up to $n-1$ issues. There is a unique Nash equilibrium when candidates maximize their vote share, and it exists for relatively small degrees of differentiation. A bridge between these strands of the literature is Soubeyran (2009), where candidates differ on their productivity in the production of two public goods and may have an absolute advantage in addition to a comparative advantage in production, compared to the other candidates.

The candidates in our paper are differentiated, but only in the sense that they have different sets of promises they can make. There is no additional dimension for quality or other traits. Further, we are interested in finding individual alternatives that are majority preferred for one candidate to anything else that the opponent can promise, rather than vote share maximizing points. In contrast to Soubeyran (2009), Krasa and Polborn (2012),and Matakos and Xefteris (2017), we consider only spatial issues exclusively, rather than public goods or redistribution.

To the best of our knowledge, the only paper which considers two candidate competition with a strategy space restriction is Asay (2008). In that model, there are three voters, and the two candidates both have linear constraints on the same dimension, and they are disjoint. In our model, we would like to consider many voters, and allow for more general constraints and higher dimensionality. Further, the solution concept in Asay (2008) is the strong point, the point with the smallest win set, while we are interested only that the win set not intersect with anything the opponent can promise, rather than its size.

3 Preliminaries

There are n dimensions of policy, so that any particular policy is a vector $x \in \mathbb{R}^n$. The set of all possible policies is $\mathbb{R}^n$. Voters have utility given by $U(x, \theta) = -||x - \theta||^2$, where $\theta \in \mathbb{R}^n$ is the ideal point of the voter. The density of voter ideal points, $f : \mathbb{R}^n \mapsto \mathbb{R}_{++}$, is continuous with full support. Voters are identified by their ideal points, so we often speak of the measure of voters on a set, rather than the measure of voter ideal points on the set. Voters vote sincerely for their most preferred alternatives.

There are two office motivated candidates, A and B. Candidate A chooses a

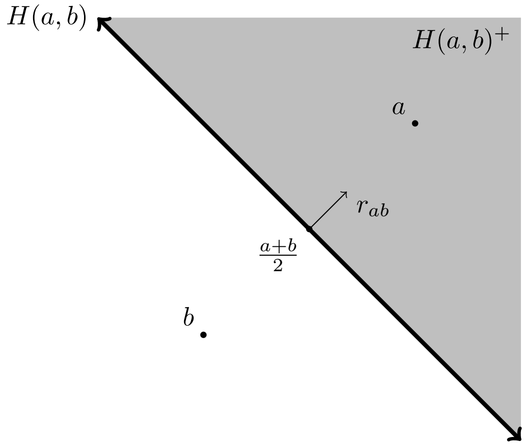

Figure 1: The hyperplane of equidistance and associated halfspaces.

policy $a \in \alpha$ and candidate B chooses a policy $b \in \beta$. *As maintained assumptions, it is assumed that both α and β are (i) non-degenerate (i.e., each of these sets contains at least two policies) and (ii) closed* .[1] Closedness ensures that for any policy of one candidate, there is a closest policy to it among the policies available to the other candidate. Additional assumptions on α and β, such as convexity, will be added as they are used in Section 4.

Consider a hyperplane H in $\mathbb{R}^n$ with normal vector in direction r containing point p. Associated with H are the following halfspaces:

$$H^+ = \{x \in \mathbb{R}^n \mid x \cdot r \geq p \cdot r\},$$

$$H^{++} = \{x \in \mathbb{R}^n \mid x \cdot r > p \cdot r\},$$

$$H^- = \{x \in \mathbb{R}^n \mid x \cdot r \leq p \cdot r\},$$

and

$$H^{--} = \{x \in \mathbb{R}^n \mid x \cdot r < p \cdot r\}.$$

The set of points equidistant to a and b is the *hyperplane of equidistance*, denoted $H(a,b)$. It is a hyperplane through the midpoint of the line connecting a and b with normal vector $r_{ab} = \frac{a-b}{||a-b||}$. Note that r_{ab} is always a unit vector. By construction, r_{ab} is always pointing from the midpoint toward a. Denote the halfspace containing a by $H(a,b)^+$, which is the set of points no further from a than b. The interior of the halfspace is denoted $H(a,b)^{++}$, which is the set of points strictly closer to a than b. An example is depicted in Figure 1.

Definition 1. The set

$$H(a,b) = \{x \in \mathbb{R}^n \mid x \cdot r_{ab} = \frac{a+b}{2} \cdot r_{ab}\}$$

is the *hyperplane of equidistance for a and b.*

[1]These assumptions are not stated explicitly in our results.

Note that for any point x, $||a - x|| \leq ||b - x|| \iff x \in H(a,b)^+$ and $||a - x|| < ||b - x|| \iff x \in H(a,b)^{++}$. A voter with ideal point θ therefore prefers a over b if and only if $\theta \in H(a,b)^{++}$.

Let $\succeq$ represent weak majority preference, and $\succ$ represent strict majority preference. Then

$$a \succeq b \iff \int_{H(a,b)^+} f(x) \geq 1/2$$

and

$$a \succ b \iff \int_{H(a,b)^+} f(x) > 1/2.$$

The set of exactly indifferent voters is a set of measure zero in $\mathbb{R}^n$, so they can be ignored.

We will carry out our analysis from the perspective of candidate A without loss of generality. All of the results will hold for candidate B with an appropriate replacement of a with b and α with β.

3.1 The Disjoint Condorcet Winner

A Condorcet Winner is a policy that is weakly majority preferred to any other policy that can be proposed.

Definition 2. A policy a is called a *Condorcet Winner (CW)* if and only if $a \succeq b, \ \forall b \in \mathbb{R}^n$.

The Condorcet Winner is an inappropriate solution concept for situations in which two candidates have different sets of policies they can promise, as it requires that a given alternative is preferred to any other alternative in the feasible policy space. We want to find a policy a that is majority preferred to any policy $b \in \beta$, rather than any policy $b \in \mathbb{R}^n$.

Definition 3. A policy a is called a *Disjoint Condorcet Winner on β (DCW)* if and only if $a \succeq b, \ \forall b \in \beta$. If $a \succ b, \ \forall b \in \beta$, then a is a *Strict Disjoint Condorcet Winner on β (SDCW).*

The following example of four alternatives and three voters illustrates the distinction between a CW and a DCW. Each individual's strict preferences over alternatives w, x, y, and z are listed in descending order. Suppose alternatives w and x are policies only candidate A can offer, and alternatives y and z are policies only candidate B can offer. For these preferences, alternative x is a DCW, but would not be a CW:

Voter J	Voter K	Voter L
x	y	w
y	z	x
w	w	y
z	x	z

Alternative x is majority preferred to both c and d, which are all B can propose. While alternative x would lose to w, since alternatives w and x are both in A's policy set, they will never be put in competition with one another.

There is some intuition for why restricting the set of alternatives has stabilizing properties. Whenever there is a preference cycle, the fact that one set of policies can only be proposed by A and the other by B eliminates any preference cycle contained entirely in α or β. This reflects the fact that a proposed policy a by A has to be weakly better than anything B can propose, not to anything anyone can propose. If there is some cycle entirely contained in α, say a_1, a_2, a_3, a_1, an equilibrium may still exist, because a_1 will never be put in a pairwise contest with a_2, because B cannot propose a_2. In fact, not only can cycles entirely in α be ignored, any cycle which contains two consecutive elements of α can be eliminated. If the only way to beat a_j is with a_k, then a_j will be a DCW, because a_k cannot be proposed by the other candidate. The only cycles that cannot be eliminated with the DCW concept are those which alternate between α and β for each element of the sequence.

When voters have Euclidean preferences and the policy space is $\mathbb{R}^n$, nonexistence of a Condorcet Winner implies a top cycle set that encompasses the entire policy space (Cohen, 1979). If restricting the admissible sets for each candidate breaks the top cycle set, then this should suggest the existence of a Disjoint Condorcet Winner. This further motivates our analysis.

3.2 The Guaranteed Supporter Set

In order for a to defeat every policy $b \in \beta$, for each b there has to be a way to find a group of at least half of the voters who prefer a to b. A starting point to identifying when this is possible is to identify the group of people who always support a over b, or a group of guaranteed supporters. We show that this group can be identified by considering only the hyperplanes of equidistance for the policies closest to a in β.

To determine who is a guaranteed supporter of a, we first consider in what directions from a policies $b \in \beta$ may lie. Since these are directions a may be challenged in, we say they are *vulnerable*.

Definition 4. A direction r is a *vulnerable direction at a* if and only if $\exists b \in \beta$ *s.t.* $b = a - \gamma r,\ \gamma \geq 0$.

Recall that directions are normalized to have a length of one. For any $b \in \beta$, there is a $\gamma \geq 0$ such that $b = a - \gamma r$ for some r, namely $r = \frac{a-b}{||a-b||}$ and $\gamma = ||a - b||$. The minus sign accounts for the fact that r is always pointing from b toward a. The collection of rs required to reach every point $b \in \beta$ is the *set of vulnerable directions*, $\omega(a, \beta)$.

Next we want to know how close a policy $b \in \beta$ can be relative to a, for each vulnerable direction. The set

$$\beta(r) \equiv \{b \in \beta | b = a - \gamma r \text{ for some } \gamma \geq 0\}$$

is the set of all policies in β that lie in direction r from a. Let $b(r)$ represent the closest policy $b \in \beta$ to a in direction r, or formally,

$$b(r) \equiv \arg\min_b \{||a - b|| \ s.t\ b \in \beta(r)\}.$$

The closedness of β ensures that $b(r)$ is always defined. See Figure 2.

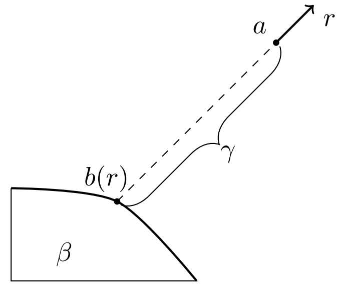

Figure 2: The closest point in β to a in direction r, $b(r)$.

For each direction $r \in \omega(a, \beta)$, we now show that if θ is weakly closer to a than $b(r)$, then it is strictly closer to a than every $b \neq b(r)$ in $\beta(r)$. Thus, if a is preferred to $b(r)$, then a will be strictly preferred to any policy b further from a in direction r.

Lemma 1. For any ideal point θ, for any policy $a \in \alpha$, any vulnerable direction r at a, and any $b \in \beta(r)$ for which $b \neq b(r)$, $||a - \theta|| \leq ||b(r) - \theta||$ implies $||a - \theta|| < ||b - \theta||$.

Consider a voter with ideal point θ and a policy a such that θ is not closer to b than a for any $b \in \beta$. Then for each direction $r \in \omega(a, \beta)$, θ must be closer to a than $b(r)$, or else that voter would prefer $b(r)$. By Lemma 1, if a is preferred to $b(r)$, then a will also be preferred to any policy b further from a in direction r. Thus, a is weakly preferred to every b in direction r by a voter with ideal point θ if and only if $\theta \in H(a, b(r))^+$. In order for this to be true for every policy $b \in \beta$, it needs to be true that $\theta \in H(a, b(r))^+$, $\forall r \in \omega(a, \beta)$. Therefore, a voter never prefers b to a if and only if $\theta \in \bigcap_{r \in \omega(a,\beta)} H(a, b(r))^+$. We call this the *Guaranteed Supporter Set (GS)*:

$$GS(a, \beta) \equiv \bigcap_{r \in \omega(a,\beta)} H(a, b(r))^+.$$

Note that the set $GS(a, \beta)$ is a closed convex set, as it is the intersection of closed halfspaces.

Propositon 1. For any policy $a \in \alpha$, an ideal point θ satisfies:

(i) $||a - \theta||^2 \leq ||b - \theta||^2, \ \forall b \in \beta$, if and only if $\theta \in GS(a, \beta)$.

(ii) $||a - \theta||^2 < ||b - \theta||^2, \ \forall b \in \beta$, if and only if $\theta \in \text{int}GS(a, \beta)$.

(iii) $||a-\theta||^2 \leq ||b-\theta||^2$, $\forall b \in \beta$ and $||a-\theta||^2 = ||b(r)-\theta||^2$ for some $r \in \omega(a, \beta)$ if and only if $\theta \in \text{bd}(GS(a, \beta))$.

Proposition 1 allows us to consider $GS(a, \beta)$ either as the intersection of halfspaces or as the set of ideal points that are closer to a than any $b \in \beta$. Both ways of defining this set are useful.

3.3 Median Hyperplanes

In this section, we introduce median hyperplanes and describe the generic conditions that need to be satisfied for a point to be a DCW, assuming only that β is closed and nonempty. The *median hyperplane in direction* r, $M(r)$, has r as a normal vector and is situated so that half of the voters lie on one side of it and half lie on the other. This hyperplane is determined by the direction r and a point $m(r) \in \mathbb{R}^n$.

Definition 5. A *median hyperplane for direction* r is the set

$$M(r) = \{x \in \mathbb{R}^n | r \cdot x = r \cdot m(r)\},$$

where $m(r) \in \mathbb{R}^n$ is chosen so that

$$\int_{M(r)^+} f(x) \geq 1/2$$

and

$$\int_{M(r)^-} f(x) \geq 1/2.$$

Note that the point $m(r)$ used to define $M(r)$ is not unique. Any other point on this hyperplane could be used instead. However, it is often convenient to choose $m(r)$ to lie on the line in the direction r from a. The median hyperplane for direction r is also coincident with the median hyperplane for direction $-r$.

Lemma 2. For any direction r,

$$M(r) = M(-r).$$

The median hyperplanes characterize the majority preference over pairs of policies. For any two policies a and b, a is majority preferred to b if and only if $M(r_{ab}) \subset H(a, b)^+$. An example is depicted in Figure 3.

Lemma 3. For any $a, b \in \mathbb{R}^n, a \succeq b$ if and only if $M(r_{ab}) \subset H(a, b)^+$, with strict preference when $M(r_{ab}) \subset H(a, b)^{++}$.

Next, we extend Lemma 3 to show that only the closest policy $b(r)$ in direction r needs to be considered.

Lemma 4. For any policy $a \in \alpha$ and any vulnerable direction r at a, a is majority preferred to every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^+$, with strict majority preference if and only if $M(r) \subset H(a, b(r))^{++}$.

To gain some geometric intuition for the conditions in Lemma 4, consider an initial situation in which $M(r) \subset H(a, b(r))^+$, as illustrated in Figure 4. Suppose that voters' preferences change so that $M(r)'$, the new median hyperplane for direction r, is a subset of $M(r)^{++}$. Initially, $r \cdot \frac{a+b(r)}{2} \leq r \cdot m(r)$. Because $M(r)' \subset M(r)^{++}$, it must be that $r \cdot m(r)' > r \cdot m(r)$. Hence, $r \cdot \frac{a+b(r)}{2} < r \cdot m(r)'$. Any weak majority preference for a over $b(r)$ is now strict, and any strict majority preference is preserved. Intuitively, a voter with ideal point $m(r)'$ prefers to *move further from* β than one with ideal point $m(r)$.

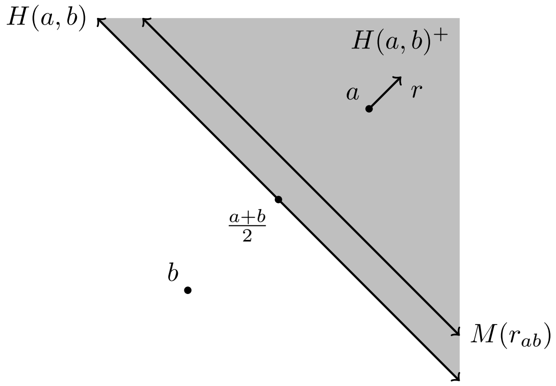

Figure 3: Policy a is strictly majority preferred to b.

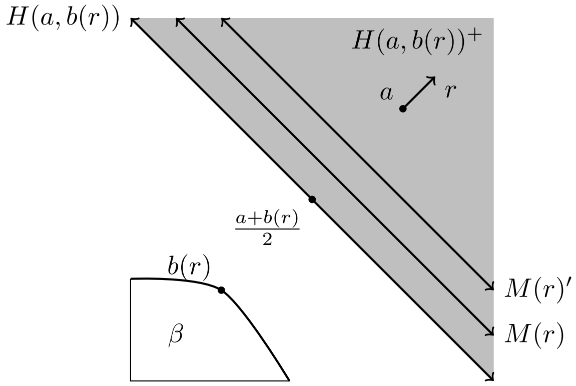

Figure 4: $M(r)'$ is strictly further from β in direction r than $M(r)$.

Definition 6. For two hyperplanes H and H' with the same normal vector r, H' is *weakly further from* β than H if and only if $H' \subset H^+$ and is *strictly further from* β than H if and only if $H' \subset H^{++}$.

As illustrated in Figure 4, if $a \succeq b(r)$, then it must be that the median hyperplane $M(r)$ is weakly further from β than $H(a, b(r))$. We can now succinctly characterize what must be true of the median hyperplanes in order for a to be a DCW or SDCW.

Propositon 2. A policy $a \in \alpha$ is a DCW if and only if $M(r)$ is weakly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$. A policy $a \in \alpha$ is an SDCW if and only if $M(r)$ is strictly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$.

4 Median Hyperplanes and Guaranteed Support

Proposition 2 identifies the circumstances in which a policy a is a DCW in terms of the relationship between the median hyperplane in direction r and the hyperplane of equidistance in this direction, for every vulnerable direction r.We now show how this characterization of a DCW can be extended to characterize a DCW in terms of the guaranteed supporter set, $GS(a, \beta)$.

A natural conjecture is that a is a DCW if and only if all median hyperplanes are supporting hyperplanes of $GS(a, \beta)$ or are strictly further from β. After all, $GS(a, \beta)$ is the intersection of all of the halfspaces of equidistance. We show in Theorem 1 that this is generically a sufficient condition. To formalize this argument, let $S(r)$ be the supporting hyperplane to $GS(a, \beta)$ with normal vector r. Since $GS(a, \beta)$, this is hyperplane is unique. The median hyperplane is supporting to $GS(a, \beta)$ if and only if $M(r) = S(r)$. Consequently, all median hyperplanes are supporting hyperplanes of $GS(a, \beta)$ or are strictly further from β if and only if $M(r)$ is weakly further from β than $S(r)$.

Theorem 1. If for every $r \in \omega(a, \beta)$, $M(r)$ is weakly further from β than $S(r)$, then a is a DCW. If for every $r \in \omega(a, \beta)$, $M(r)$ is strictly further from β than $S(r)$, then a is an SCDW.

However, it is not generically necessary for all median hyperplanes to support $GS(a, \beta)$. Difficulty arises when there are kinks in the boundary of $GS(a, \beta)$, as illustrated in Figure 5. A kink implies that some of the hyperplanes of equidistance may not support $GS(a, \beta)$ because the other constraints have effectively made them redundant. In the case of Figure 5, where $\beta = \{b_1, b_2, b_3\}$, b_1 is so far away from a that an ideal point that is closer to a than b_2 or b_3 is necessarily closer to a than b_1. At the kink point, $x \in H(a, b_2)$ and $x \in H(a, b_3)$, but $x \in H(a, b_1)^{++}$. As a consequence, the median line for r_{ab_1} could be located between the kink point of $GS(a, \beta)$ and $H(a, b_1)$ and still imply that a is a DCW.

However, at any point on the boundary of $GS(a, \beta)$ that is uniquely supported by a hyperplane H with normal vector r, it is in fact necessary that the median hyperplane $M(r)$ either coincides with H or is further from β in this direction than it.

Lemma 5. If $a \in \alpha$ is a DCW and there exists $S(r)$ that uniquely supports a point $x^* \in GS(a, \beta)$, then it is necessary that $M(r)$ is weakly further from β than $S(r)$.

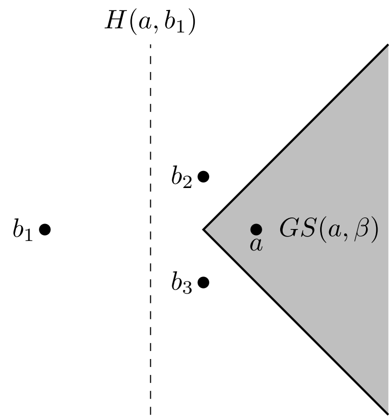

Figure 5: In this example, $\beta = \{b_1, b_2, b_3\}$. Hyperplanes $H(a, b_2)$ and $H(a, b_3)$ support $GS(a, \beta)$, while $H(a, b_1)$ does not.

A kink point on the boundary of $GS(a, \beta)$ is necessarily equidistant to multiple policies in β. However, if β is closed and convex, we know that there must be a unique policy $b \in \beta$ that is closest to any policy $x \notin \beta$. As a consequence, there cannot be kinks on the boundary of $GS(a, \beta)$ if β is assumed to be convex in addition to our maintained assumption that it is closed.

Lemma 6. If β is convex, $GS(a, \beta)$ is uniquely supported at every point in $\text{bd}(GS(a, \beta))$.

It need not be the case that there is a solution to $\min_{x \in GS(a,\beta)} r \cdot x$ for every $r \in \omega(a, \beta)$. However, if there is and β is convex, then a necessary and sufficient condition for a to be a DCW is that all median hyperplanes are either supporting to $GS(a, \beta)$ or are further from β in direction r.

Theorem 2. If β is convex and $\min_{x \in GS(a,\beta)} r \cdot x$ has a solution for every $r \in \omega(a, \beta)$, then a is a DCW if and only if $M(r)$ is weakly further from β than $S(r)$ for every $r \in \omega(a, \beta)$.

By the Weierstrass Theorem, the assumption in Theorem 2 that the minimization problem has a solution for every $r \in \omega(a, \beta)$ is satisfied if $GS(a, \beta)$ is compact. As an example, consider the classic spatial model, in which $\beta = \mathbb{R}^n$. For any proposed DCW a, which will be equivalent to a CW in this context, the closest point in β in each direction r will be a for every direction. The only ideal point which can be weakly closer to a than every $b \in \beta$ is $\theta = a$, so $GS(a, \beta) = \{a\}$. Since $\mathbb{R}^n$ is convex and $\{a\}$ is compact, the conditions of Theorem 2 are met, and therefore $M(r)$ must support $\{a\}$ for every direction r. In other words, a must be a total median. This fact is formally stated in the Corollary 1.

Corollary 1. If $\beta = \mathbb{R}^n$, then a is a DCW if and only if a is a total median.

Theorem 2 establishes that the guaranteed supporter set, under some regularity conditions, characterizes the conditions for a to be a DCW. Informally, the requirement that each $M(r)$ either support $GS(a,\beta)$ or lie further from β is a requirement that the median voter for direction r can be chosen to be a guaranteed supporter of a, or to have a stronger preference for a over $b(r)$.

4.1 Comparison to Classic Spatial Model: Conditions for an Equilibrium

Theorem 2 provides conditions on the median hyperplanes in order for a to be a DCW. It does not, however, give any insight into when these conditions are weaker than those in the classic spatial model.

Consider any policy a and direction $r \in \omega(a,\beta)$. Generically, by Lemma 4, it is necessary in order for a to be a DCW that $M(r) \subset H(a,b(r))^+$. Further, because $M(r) = M(-r)$ by Lemma 2, it must also be the case that $M(r) \subset H(a,b(-r))^+$, if $-r$ is also vulnerable. Therefore, if both r and $-r$ are vulnerable directions, a can be a DCW only if

$$M(r) \subset H(a,b(r))^+ \cap H(a,b(-r))^+$$

or equivalently, if

$$r \cdot \frac{a+b(r)}{2} \leq r \cdot m(r) \leq r \cdot \frac{a+b(-r)}{2}.$$

In the classic spatial model, the closest point in β in each direction is a. In this case, the set that $M(r)$ must lie in becomes degenerate, because $M(r)$ must satisfy

$$r \cdot \frac{a+a}{2} \leq r \cdot m(r) \leq r \cdot \frac{a+a}{2},$$

or equivalently,

$$r \cdot a \leq r \cdot m(r) \leq r \cdot a.$$

The only possible choice of $M(r)$ is one that passes through a. However, we will show in Theorem 3, if $a \notin \beta$, and both r and $-r$ are vulnerable directions, then it cannot be the case that $b(r) = b(-r)$ or that either is equal to a, and hence $H(a,b(r))^+ \cap H(a,b(-r))^+$ is no longer degenerate. There is a positive distance between the hyperplanes of equidistance, it is only necessary that $M(r)$ be located somewhere in the space between them. While it is sufficient for $M(r)$ to pass through a, this need not be the case. We therefore do not require a total median, even if all directions are vulnerable.

Theorem 3. If $a \notin \beta$ and all directions are vulnerable, then a is a *DCW* if and only if $\forall r$, $M(r) \in H(a,b(r))^+ \cap H(a,b(-r))^+$, a set containing a with a nonempty interior.

We now show if $a \notin \beta$ and β is convex, then at most half of the possible directions are vulnerable. In particular, if a given direction r is vulnerable, then $-r$ cannot be vulnerable as well; otherwise, a lies on a line connecting two policies in β and must therefore be in β.

Lemma 7. If $a \notin \beta$ and β is convex, then $r \in \omega(a,\beta) \implies -r \notin \omega(a,\beta)$.

When r is a vulnerable direction but $-r$ is not, then $b(-r)$ does not exist. Consequently, the requirement for a to be a DCW is that $M(r) \subset H(a, b(r))^+$ by Lemma 4. There is no longer a second constraint on $M(r)$ to consider. Therefore, if in addition to $a \notin \beta$, it is also assumed that β is convex, the conditions for existence of a DCW are weakened further in two ways: (i) there are fewer directions that are vulnerable and (ii) for each direction that is vulnerable, the requirement is that the median hyperplane for that direction lies in a given halfspace; it need not pass through a specific point or subset of that halfspace. This is stated formally in Theorem 4.

Theorem 4. If $a \notin \beta$ and β is convex, then at most half of all directions are vulnerable, and further, a is a DCW if and only if for each vulnerable direction, $M(r) \subset H(a, b(r))^+$.

The difference in the conditions for equilibrium in each of the three cases discussed are summarized in Table 1.

Assumptions	Vulnerable directions	Median Constraint for each vulnerable direction
$\beta = \mathbb{R}^n$	All	$M(r)$ must pass through a
$a \notin \beta$	All	$M(r)$ must lie in a set with nonempty interior which contains a.
$a \notin \beta$ and β is convex	At most half	$M(r)$ must lie in a half- space containing a

Table 1: A summary of the conditions for a to be a DCW.

Note that Theorem 4 rests upon $a \notin \beta$, rather than $\alpha \bigcap \beta = \emptyset$. If we chose $a \in \text{int}(\beta)$, then $b(r) = b(-r) = a$ for all directions, so $GS(a, \beta)$ would remain a singleton, and a total median is necessary for a to be a Condorcet Winner. If we chose $a \in \text{bd}(\beta)$, then we would require an "almost" total median, in which all median lines would need to either support a or be further in their respective directions. These kinds of situations are explored in detail in Matthews (1980), and so we do not repeat that analysis here. The results in this paper pertain to the novel case in which there exists $a \in \alpha$ such that $a \notin \beta$.

4.2 Comparison to Classic Spatial Model: Existence of any Equilibrium

Aside from the question of whether a given policy a is a DCW, we may also want to know under what conditions there exists any a in α such that a is a DCW. To characterize these conditions, we use of the *yolk*. The yolk is the smallest hypersphere that is either tangent to or intersects all median hyperplanes (McKelvey, 1986). Formally, the yolk is determined by a choice of a center $c \in \mathbb{R}^n$ and a radius $\tau \in \mathbb{R}_+$, so that $Y(c, \tau) \equiv \{y \in \mathbb{R}^n | \; ||y - c||^2 = \tau^2\}$ satisfies $Y(c, \tau) \cap M(r) \neq \emptyset$, $\forall r \in \omega(a, \beta)$. A total median corresponds to a yolk with $\tau = 0$.

Whenever $Y(c,\tau)$ is contained entirely in $GS(a,\beta)$, then a is a DCW, because all median hyperplanes must either be supporting $GS(a,\beta)$ or strictly further from β. If the yolk is contained in the interior of $GS(a,\beta)$, then a is an SDCW.

Lemma 8. Whenever $Y(c,\tau) \subset GS(a,\beta)$, a is a DCW, and whenever $Y(c,\tau) \subset \text{int}GS(a,\beta)$, a is an SDCW.

It is not possible to derive a necessary and sufficient condition for the existence of a DCW using the yolk, since it does not place a precise restriction on the location of all median hyperplanes, and it considers many directions that may not be vulnerable at a given $a \in \alpha$. The directions that will be vulnerable are determined by the specific location of a, so it is difficult to make any prior assumption about which directions can be ignored when considering existence of a DCW. However, generically in spatial voting in $\mathbb{R}^n$, if $\tau > 0$, there is no equilibrium, no matter where the yolk is located. In this model, we can have $\tau > 0$ and still have an equilibrium, when $Y(c,\tau)$ lies within $GS(a,\beta)$ for at least one $a \in \alpha$ or within $GS(b,\alpha)$ for at least one $b \in \beta$. By Proposition 1, the yolk is contained in $GS(a,\beta)$ for at least one a in α if there exists an $a \in \alpha$ such that $||y-a|| < ||y-b^*(y)||$ for every $y \in Y(c,\tau)$, where $b^*(y)$ is the closest point to y in β. This must certainly hold if $\max_{y\in Y(c,\tau)} ||y-a|| < \min_{y\in Y(c,\tau)} ||y-b(y)||$. The furthest a point in the yolk can be from a is $||c-a|| + \tau$. The closest a point in the yolk can be to a point in β is $||c-b(c)|| - \tau$.

$$||c-a|| + \tau < ||c-b^*(c)|| - \tau$$

or equivalently,

$$||c-a|| < ||c-b^*(c)|| - 2\tau$$

Candidate A can minimize the left hand side of this inequality by choosing $a = a^*(c)$, the closest point to c in α. Intuitively, this says that if one candidate can get at least 2τ closer to the center of the yolk than their opponent, then there is a DCW in that candidate's policy set. For cases in which this doesn't hold, it is still possible that there exists a DCW is α, as this is only a sufficient condition. The preceding argument establishes the following theorem:

Theorem 5. Given $Y(c,\tau)$, a DCW exists if

$$||c-a^*(c)|| < ||c-b^*(c)|| - 2\tau$$

or

$$||c-a^*(c)|| > ||c-b^*(c)|| - 2\tau.$$

We may gain some intuition by considering what happens when $\tau = 0$. In this case, we can think of c as the ideal point of a decisive voter, and whichever candidate is capable of making that voter strictly better off is the winner of the election. This reflects the intuition from the one dimensional case, in which the winner is the candidate who makes the median voter better off. When $\tau > 0$, if one candidate can take a position 2τ closer to c than the opponent, then she wins for sure. In a coarse way, the intuition that candidates can win by appealing to a centrally located voter is maintained.

5 Response of Median Hyperplanes to perturbations of the distribution

An equilibrium in the classic spatial model is extremely fragile. Changing voters' ideal points even infinitesimally can cause the equilibrium to no longer exist (Rubinstein, 1979). In this section, we show that equilibria in our model are robust to two types of changes in the distribution of the voters. First, in Theorem 6, we show that if a is a DCW, there are some directions in which voters can be moved by an arbitrary distance while preserving a as a DCW. Second, in Theorem 7, we show that if a is an SDCW, we can move voters in arbitrary directions by a small distance while preserving a as an SDCW. In this model, not only does a DCW continue to exist when voters are moved, but it is the same policy as before.

To prove these theorems, we first establish in Proposition 3 in what direction and by what magnitude median hyperplanes move when some of the density of voters is moved in a direction r^*. We can then determine what movements of voters will preserve a particular point as a DCW or an SDCW.

A distribution of voters f' formed by moving density in the direction r^* relative to f will have median hyperplanes that are further from β for any direction r such that $r \cdot r^* \geq 0$. To get the intuition, consider the case when voter ideal points are discrete. If a voter is moved in direction r^* by magnitude $\delta > 0$, the projection of the voter's ideal point onto direction r is non-decreasing for any r such that $r \cdot r^* \geq 0$. By definition,

$$\theta \in H(a, b(r))^+ \iff \theta \cdot r \geq \frac{a + b(r)}{2} \cdot r.$$

Thus for an ideal point shifted by δ in direction r^*,

$$(\theta + \delta r^*) \cdot r = \theta \cdot r + \delta r^* \cdot r \geq \frac{a + b(r)}{2} \cdot r,$$

because $r^* \cdot r \geq 0$. Hence, if that voter previously had an ideal point in $H(a, b(r))^+$, the voter's new ideal point will also be in $H(a, b(r))^+$. Therefore, if there was majority preference for a over $b(r)$ when voters are distributed according to f, then that preference must remain, because any voter that supported in the original distribution a over $b(r)$ does so in the new distribution as well. In the continuous case, rather than moving individual voters, we reduce the density of voters on a set of positive measure and increase it on another set, but by the same intuition, majority preference for a over $b(r)$ will be maintained.

For a direction r such that $r \cdot r^* < 0$, it may be the case that the median hyperplane moves toward β. However, the furthest the median hyperplane could move is the distance δ the voter has been moved. If a hyperplane intersects the moved voter's ideal point, assuming it is necessary for the median hyperplane to move at all, then it is certain that at least half of the voters are in the positive halfspace of this new hyperplane. If there are any ideal points located between the hyperplane and the original median hyperplane, then there will be strictly more than half of the voters in its positive halfspace. Thus, the furthest the median hyperplane must move is δ.

To carry out this transformation in the continuous case, we pick a two sets of positive measure in $\mathbb{R}^n$, Γ and Γ', such that Γ' is a linear transformation of

Γ in direction r^* by some distance δ. Then, we reduce the density on the set Γ by a positive ϵ, and increase the density on Γ' by ϵ.

Definition 7. Let Γ be a set of positive measure in $\mathbb{R}^n$. Let $\Gamma' = \Gamma + \delta r^*$, where $\delta > 0$ and $r^* \in \mathbb{R}^n$ is a unit vector. Choose $\epsilon > 0$ such that $\epsilon < f(x),\ \forall x \in \Gamma$. Let

$$f(x)' = \begin{cases} f(x) - \epsilon & \text{if } x \in \Gamma \\ f(x) + \epsilon & \text{if } x \in \Gamma' \\ f(x) & \text{if } x \notin \Gamma \cup \Gamma'. \end{cases}$$

We first prove that f' is strictly positive on $\mathbb{R}^n$ and that the integral of f' over $\mathbb{R}^n$ is one, so that f' is a probability density function.

Lemma 9. The function f' is everywhere strictly positive, and $\int_{\mathbb{R}^n} f(x)' = 1$. Therefore, f' is a probability density function.

Let $\{M(r) - dr\} \equiv \{x \in \mathbb{R}^n | r \cdot x = r \cdot m(r) - d\}$ be a shift of the median hyperplane $M(r)$ in the direction $-r$ by d units. At $d = 0$, this is equal to $M(r)$. Consider the integral over its positive halfspace, $\int_{\{M(r)-dr\}^+} f(x)'$. As d increases, this hyperplane moves further in direction $-r$, expanding the set $\{M(r) - dr\}^+$ to include more of the measure of voters. Thus, the integral is strictly increasing in d.

Lemma 10. The value of $\int_{\{M(r)-dr\}^+} f(x)'$ is strictly increasing in d.

Let $M(r)'$ be the median hyperplane for direction r when voter ideal points are distributed according to f'. Because $M(r)$ and $M(r)'$ have the same normal vector, there must exist d such that $M(r)' = M(r) - dr$. Hence, $M(r)' \subset M(r)^+$ as long as the value of d is not positive. A positive value of d indicates that $M(r)'$ is not weakly further from β than $M(r)$. However, we show in Lemma 11 that this can only happen if the change in the distribution from f to f' caused $\int_{M(r)^+} f(x)' < \frac{1}{2}$.

Lemma 11.

$$M(r)' \not\subset M(r)^+ \iff \int_{M(r)^+} f(x)' < \int_{M(r)^+} f(x).$$

For any r such that $r \cdot r^* \geq 0$, there is either no change in the density of voters in $M(r)^+$ or an increase, so by Lemma 11, the median hyperplane $M(r)'$ is weakly further from β than $M(r)$. For any r such that $r \cdot r^* < 0$, it is possible that there is less density of voters in $M(r)^+$ when ideal points are distributed f', so it may be the case that $M(r)' \not\subset M(r)^+$. However, we can bound the distance of $M(r)'$ from $M(r)$ by δ.

Propositon 3. For any direction r for which $r \cdot r^* \geq 0$, $M(r)' \subset M(r)^+$. For any direction r for which $r \cdot r^* < 0$, the distance between $M(r)'$ and $M(r)$ is at most δ.

Consider a policy a and let $\omega(a, \beta)^+ = \{k \in \mathbb{R}^n | k \cdot r \geq 0, \forall r \in \omega(a, \beta)\}$. By Proposition 3, for any $r^* \in \omega(a, \beta)^+$, if any subset of voters is moved in the direction r^* by any distance, then the median hyperplanes for every vulnerable direction would either remain the same or move further from β. Hence, if a is a DCW for the original distribution, it is a DCW for the new distribution as well.

Theorem 6. Let a be a DCW when ideal points are distributed according to $f(x)$. Consider a change of the density of voter ideal points from f to f', with $r^* \in \omega(a, \beta)^+$. Then a is a DCW when voters have ideal points distributed according to f'.

If policy a is an SDCW for distribution f, then $M(r) \subset H(a, b(r))^{++}$ for every vulnerable direction. Consider a distribution f' formed by a movement of any subset of the voters in any direction r^*. Since the median hyperplanes move by at most δ for any direction for which $r \cdot r^* < 0$, by making δ sufficiently small, it is ensured that $M(r)' \subset H(a, b(r))^{++}$. Hence, a is an SDCW for f' as well.

Theorem 7. Let a be an SDCW when ideal points are distributed according to f. Consider a change of the density of voter ideal points from f to f', with $r^* \notin \omega(a, \beta)^+$ and δ sufficiently small. Then a is an SDCW when voters have ideal points distributed according to f'.

Theorems 6 and 7 show that equilibrium in this model is more robust than in the classic spatial model. Since existence of an equilibrium in the classic model depends upon the existence of a total median, moving even a small amount of mass an infinitesimal distance can cause the equilibrium to no longer exist. In this model, because candidate B is limited in what policies she can propose, moving voters away from the policies B can offer and toward a will never reduce support for a. Further, even if voter opinion moves marginally away from a and toward β in some directions, if a was winning with a strict majority over any position B could offer, while that majority may be reduced, a may still be a winning position.

However, Theorems 6 and 7 also imply that the DCW need not coincide with any measure of centrality, be it the mean, the center of the yolk, or a total median if one exists. Suppose policy a is a DCW. Even if a were initially coincident with some measure of centrality, by Theorem 6, we can perturb the distribution in a way that moves the center of the yolk, the mean, or the total median if one exists, and a will remain a DCW. This is in contrast to the classic spatial model, in which an equilibrium, if it should exist, must be located at the total median. Further, candidates in this model need not be responsive to changes in voter opinion. Policy a may remain a DCW even if voter ideal points move, and in Theorem 6, they may move an arbitrary distance without disturbing a as an equilibrium. This implies that stability may come at a cost of reduced responsiveness.

6 Application: Two Party Competition

To illustrate the guaranteed supporter set and its implications, we return to the example from the introduction. Suppose that candidates are constrained by their respective parties to take positions opposite one another on both issues: gun rights , x ,and defense spending , y. Without loss of generality, we will assume that candidate A, the Republican, is restricted to non-negative positions on each of the two issues, while candidate B , the Democrat, is restricted to non-positive positions on each of the two issues. Therefore, $\alpha \equiv \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\}$ and $\beta \equiv \{(x, y) \in \mathbb{R}^2 | x \leq 0, y \leq 0\}$. These policy sets correspond to the nonnegative and nonpositive quadrants respectively, as shown in Figure 6.

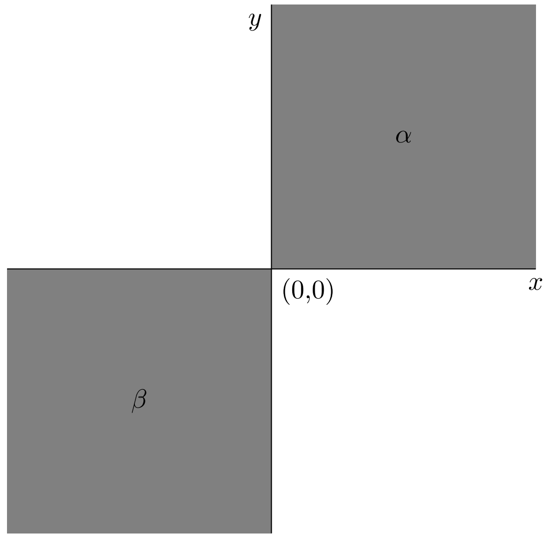

Figure 6: Policy sets for A and B in two party competition.

To derive the guaranteed supporter set for a given position a, we use the characterization in Proposition 1, and look for voters that are weakly closer to a than the closest policy in β. For voters in Quadrant I, the closest possible policy in β is the origin. Therefore, a voter is closer to a than the closest possible $b \in \beta$, and thereby a guaranteed supporter for a, if and only if he is above the hyperplane of equidistance for a and the origin. For a voter in Quadrant II, the closest policy to him in β is a policy b with the same x coordinate, located on the x axis. Therefore locus of equidistant points will be a parabola with a as its focus and the x axis as its directrix. A voter in Quadrant II will only be closer to a than the closest point in β if he lies on or above this parabola. For a voter in quadrant IV, the closest point to him in β will be the policy b with the same y coordinate, located on the y axis. Therefore the locus of equidistant points will be a parabola with a as its focus and the y axis as its directrix, and so the locus of equidistant points there will be a parabola with a as its focus and the y axis as its directrix. Hence, a voter in quadrant IV is closer a than the closest b in β to him if and only if he is located above this parabola. This results in the $GS(a, \beta)$ that appears in Figure 7.

Using Theorem 5, a DCW will exist with A as the winner in this application if there exists a policy $a \in \alpha$ that is closer to the center of the yolk c by at least 2τ than the closest $b \in \beta$ to c. Whenever c is in quadrant I, the closest $a \in \alpha$ is simply $a = c$. Therefore, there is a DCW for A if c is at least 2τ from the origin. Whenever c is in Quadrant II, the minimum distance to a policy $a \in \alpha$ is the negative of the x coordinate of c, and the minimum distance to a policy $b \in \beta$ is the y coordinate of c. Therefore, A wins if c is above the line $y = -x + 2\tau$. In Quadrant IV, a parallel argument implies the same line. A symmetric argument using Theorem 5 shows that there is a DCW for B if c is under the line $x - 2\tau = y$. Hence, whenever the center of the yolk lies in one of two halfspaces, there is a DCW for either A or B. This is illustrated by the

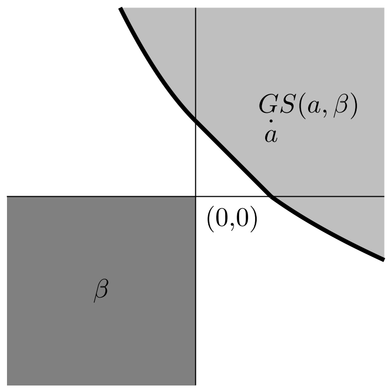

Figure 7: The GS set for a given point a.

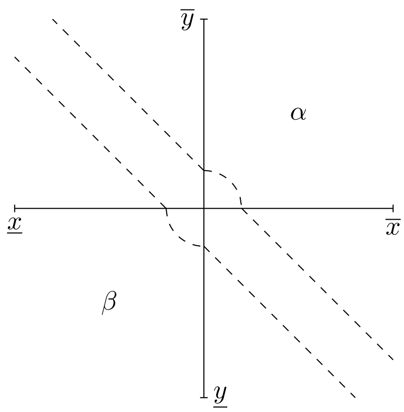

Figure 8: If c is located above the upper dashed line, there exists $a \in \alpha$ such that a is a DCW, and similarly for B when c is under the lower dashed line.

dashed lines in Figure 8. Even when the yolk is in the excluded area between them, there may still be an equilibrium, as Theorem 5 only provides a sufficient condition for the yolk to be contained in $GS(a, \beta)$, and that itself is only a sufficient condition.

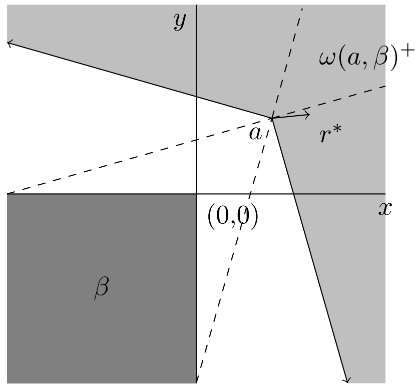

Figure 9: Policy sets for A and B in two party competition.

We can also consider the application of Theorem 6. For any DCW $a \in \alpha$, the set of vulnerable directions is always a subset of the set $\{k \in \mathbb{R}^2_+ | ||k|| = 1\}$. Consider r^* such that $r^* \in \mathbb{R}^2_+$. This implies voters are moving toward more republican positions on both gun rights and defense spending, at least weakly. As illustrated in Figure (something), such r^* will lie in $\omega(a, \beta)^+$. Therefore, by Theorem 6, any subset of the voters can be moved by any distance in direction r^*, and a will remain a DCW. The implication that is, within this model, a move in voter opinion toward more Republican positions will not decrease the support for any given Republican policy. This in contrast to the spatial model, in which both candidates would be forced to move to the new total median, should one exist, in order to avoid losing with certainty.

7 Conclusion

I have no idea how to write a good conclusion, so I will instead just list what I think the main points of the paper are for now.

- We can characterize the necessary conditions for equilibrium by a set of guaranteed supporters, and this nests the classic model.
- The conditions for equilibrium that are implied are much weaker, and allow for equilibrium to exist in more cases.
- The equilibrium in this model is more robust
- Applied to a standard way of thinking about US parties, we get intuitive outcomes.
- An interesting extension would be to see if the notion of guaranteed support is useful when preferences have more general form.

Appendix

Proof of Lemma 1. Let γ^* solve $b(r) = a - \gamma r$. Because $||a - \theta|| \leq ||b(r) - \theta||$, $a \in H(a, b(r))^+$. Therefore, $r \cdot \theta \geq r \cdot \frac{a+a-\gamma^* r}{2}$ or equivalently, $r \cdot \theta \geq r \cdot a - \frac{\gamma^*}{2}$, because r has unit length. For any $b \in \beta(r)$ such that $b \neq b(r)$, the associated γ is necessarily larger than γ^*. Because $r \cdot \theta \geq r \cdot a - \frac{\gamma^*}{2} \implies r \cdot \theta > r \cdot a - \frac{\gamma}{2}$,it must be that $\theta \in H(a, b)^{++}$. Hence, θ is strictly closer to a than b for every $b \in \beta(r)$ for which $b \neq b(r)$. □

Proof of Proposition 1. (i) If $\theta \in GS(a, \beta)$ then $\theta \in H(a, b(r))^+$, $\forall r \in \omega(a, \beta)$. Therefore, by the application of Lemma 1 for every $r \in \omega(a, \beta)$, $||a - \theta|| \leq ||b - \theta||$, $\forall b \in \beta$. If $\theta \notin GS(a, \beta)$,then there exists an r such that $\theta \notin H(a, b(r))^+$, which implies for this r, $||a - \theta|| > ||b(r) - \theta||$.

We now prove the sufficiency parts of (ii) and (iii).

(ii) If $\theta \in \text{int}GS(a, \beta)$, then $\theta \in H(a, b(r))^{++}$ and therefore $||a - \theta|| < ||b(r) - \theta||$, $\forall r \in \omega(a, \beta)$. Thus, by Lemma 1, $||a - \theta|| < ||b - \theta||$, $\forall b \in \beta$. [2]

(iii) If $\theta \in \text{bd}(GS(a, \beta))$, then by (i) , $||a - \theta||^2 \leq ||b - \theta||^2$, $\forall b \in \beta$. We now show that it must be the case that $\theta \in H(a, b(r))$ for at least one r. Suppose, contrary to our claim, that $\theta \in H(a, b(r))^{++}$, $\forall r \in \omega(a, \beta)$. Then $r \cdot \theta > r \cdot \frac{a+b(r)}{2}$, $\forall r \in \omega(a, \beta)$. Choose an $\epsilon > 0$ sufficiently small, and consider any point x within ϵ of θ. Let the direction pointing from θ to x be r'. Note that for x,

$$r \cdot x = r \cdot (\theta + \epsilon r') = r \cdot \theta + \epsilon r \cdot r' \geq r \cdot \theta - \epsilon, \ \forall r \in \omega(a, \beta)$$

because $r \cdot r' \geq -1$. Thus, for a sufficiently small epsilon,[3]

$$r \cdot \theta - \epsilon > r \cdot \frac{a + b(r)}{2}, \ \forall r \in \omega(a, \beta)$$

so that $x \in \theta \in H(a, b(r))^{++}$, $\forall r \in \omega(a, \beta)$. Since this holds for an arbitrary x within ϵ of θ, we have shown that an open ball around θ is contained in $GS(a, \beta)$, contradicting our assumption that θ is a boundary point. Hence, $\theta \in H(a, b(r))$ for at least one $r \in \omega(a, \beta)$, and thus, $||a - \theta|| = ||b(r) - \theta||$ for that r.

We now prove the necessary parts of (ii) and (iii).

(ii) Suppose $\theta \notin \text{int}GS(a, \beta)$. Then either $\theta \notin GS(a, \beta)$ or $\theta \in \text{bd}(GS(a, \beta))$. If $\theta \notin GS(a, \beta)$, then there must be at least one vulnerable direction r for which $\theta \notin H(a, b(r))^+$, and therefore $||b(r) - \theta|| < ||a - \theta||$. If $\theta \in \text{bd}(GS(a, \beta))$, then the proof of sufficiency for (iii) proves that $||a - \theta|| = ||b(r) - \theta||$ for some $r \in \omega(a, \beta)$. Thus, if $\theta \notin \text{int}GS(a, \beta)$, $||a - \theta||^2 \geq ||b - \theta||^2$ for at least one $b \in \beta$.

(iii) Consider an ideal point θ such that $||a - \theta|| = ||b - \theta||$ for some $b(r)$ and $||a - \theta||^2 \leq ||b - \theta||^2$, $\forall b \in \beta$. Clearly, $\theta \in GS(a, \beta)$ by the proof of (i). Notice that $\theta \notin \text{int}GS(a, \beta)$, because that is only possible if $||a - \theta||^2 < ||b - \theta||^2$, $\forall b \in \beta$ by (ii). Since $GS(a, \beta)$ is a closed set, θ must therefore be in the boundary. □

[2]The interior of of an infinite intersection is only a subset of the intersection of the interiors- they are not necessarily equal. This argument relies only on $\text{int}GS(a, \beta) \subset \cap_{r \in \omega} H(a, b(r))^{++}$.

[3]Namely, $\epsilon < \min_{r \in \omega(a,\beta)} r \cdot \theta - r \cdot \frac{a+b(r)}{2}$

Proof of Lemma 2. Define $H \equiv \{x \in \mathbb{R}^n | -r \cdot x = -r \cdot m(r)\}$, a hyperplane through $m(r)$ with normal vector $-r$. Observe that H and $M(r)$ are coincident:

$$\{x \in \mathbb{R}^n | -r \cdot x = -r \cdot m(r)\} = \{x \in \mathbb{R}^n | r \cdot x = r \cdot m(r)\} = M(r).$$

Also observe that

$$H^+ = \{x \in \mathbb{R}^n | -r \cdot x \geq -r \cdot m(r)\} = \{x \in \mathbb{R}^n | r \cdot x \leq r \cdot m(r)\} = M(r)^-$$

and

$$H^- = \{x \in \mathbb{R}^n | -r \cdot x \leq -r \cdot m(r)\} = \{x \in \mathbb{R}^n | r \cdot x \geq r \cdot m(r)\} = M(r)^+.$$

Consequently, it must be the case that

$$\int_{H^+} f(x) \geq 1/2$$

and

$$\int_{H^-} f(x) \geq 1/2.$$

Therefore, H is the median hyperplane for direction $-r$, and is coincident with $M(r)$. □

Proof of Lemma 3. Policy a is majority preferred to b if and only if

$$\int_{H(a,b)^+} f(x) \geq 1/2.$$

If $M(r_{ab}) \subset H(a,b)^+$, because at least half of the measure of voters is in $M(r_{ab})^+$, at least half of the measure of voters is in $H(a,b)^+$, so $a \succeq b$. Because the distribution of ideal points has a positive density on all of $\mathbb{R}^n$, if $M(r_{ab}) \subset H(a,b)^{++}$, then there must be a positive measure of voters between $M(r_{ab})$ and $H(a,b)$ and, hence, strictly more than half of the measure of voters must be in $H(a,b)^+$. Finally, if $M(r_{ab}) \subset H(a,b)^-$, then it must be that at least half of the measure of voters is closer to b than a, so $b \succeq a$, whereas if $M(r_{ab}) \subset H(a,b)^{--}$, then a symmetric argument to the case of $H(a,b)^{++}$ implies $b \succ a$. Because this exhausts all possible locations of $M(r_{ab})$, the result is shown. □

Proof of Lemma 4. For weak preference, we have $a \succeq b(r) \iff M(r) \subset H(a,b(r))^+$ by Lemma 3. Also, $M(r) \subset H(a,b(r))^+ \iff r \cdot \frac{a+b(r)}{2} \leq r \cdot m(r)$. For any $b \in \beta(r)$ such that $b \neq b(r)$, $r \cdot \frac{a+b}{2} < r \cdot \frac{a+b(r)}{2}$, and therefore $r \cdot \frac{a+b}{2} < r \cdot m(r)$. Thus, $M(r) \subset H(a,b)^{++}$, so that $a \succ b$. Hence, $M(r) \subset H(a,b(r))^+ \implies a \succeq b, \forall b \in \beta(r)$. Conversely, if $M(r) \not\subset H(a,b(r))^+$, then $a \not\succeq b(r)$ by Lemma 3. Therefore, $a \succeq b, \forall b \in \beta(r) \iff M(r) \subset H(a,b(r))^+$.

For strict preference, $a \succ b(r) \iff M(r) \subset H(a,b(r))^{++}$ by Lemma 3. For any $b \in \beta(r)$ such that $b \neq b(r)$, $r \cdot \frac{a+b}{2} < r \cdot \frac{a+b(r)}{2}$, and therefore $r \cdot \frac{a+b}{2} < r \cdot m(r)$. Thus, $M(r) \subset H(a,b)^{++}$, so that $a \succ b$. Hence, $M(r) \subset H(a,b(r))^{++} \implies a \succ b, \forall b \in \beta(r)$. Conversely, if $M(r) \not\subset H(a,b(r))^{++}$, then $a \not\succ b(r)$ by Lemma 3. Therefore, $a \succ b, \forall b \in \beta(r) \iff M(r) \subset H(a,b(r))^+$. □

Proof of Proposition 2. By Lemma 4, $a \succeq b$ for every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^+$. Therefore, $a \succeq b$ for every $b \in \beta(r)$ if and only if $M(r)$ is weakly further from β than $H(a, b(r))$. Hence, $a \succeq b$ for every $b \in \beta$ if and only if $M(r)$ is weakly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$.

Further, by Lemma 4, $a \succ b$ for every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^{++}$. Therefore, $a \succ b$ for every $b \in \beta(r)$ if and only if $M(r)$ is strictly further from β than $H(a, b(r))$. Hence, $a \succ b$ for every $b \in \beta$ if and only if $M(r)$ is strictly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$. □

Proof of Theorem 1. For every $r \in \omega(a, \beta)$, $GS(a, \beta) \subseteq H(a, b(r))^+$. Therefore, $r \cdot x \geq r \cdot \frac{a+b(r)}{2}$ $, \forall x \in GS(a, \beta)$. Hence, if $M(r) \subset S(r)^+$ then $r \cdot x \geq r \cdot \frac{a+b(r)}{2}$ $, \forall x \in M(r)$, which implies $M(r) \subset H(a, b(r))^+$. Because this holds for every $r \in \omega(a, \beta)$, by Proposition 2, a is a DCW.

If $M(r) \subset S(r)^{++}$ for every $r \in \omega(a, \beta)$, then $r \cdot x > r \cdot \frac{a+b(r)}{2}$ $, \forall x \in M(r)$, which implies $M(r) \subset H(a, b(r))^{++}$. Because this holds for every $r \in \omega(a, \beta)$, by Proposition 2, a is an SDCW. □

Proof of Lemma 5. By the definition of a supporting hyperplane, x^* must be a solution to:

$$\min r \cdot x \text{ subject to } x \in GS(a, \beta).$$

Suppose, contrary to our claim, that $r \cdot x^* > r \cdot \frac{a+b(r)}{2}$. By assumption, $S(r)$ is the unique supporting hyperplane to $GS(a, \beta)$ at x^*, so no hyperplane of equidistance can be supporting. Therefore, $r' \cdot x^* > r' \cdot \frac{a+b(r')}{2}, \forall r' \in \omega(a, \beta)$.

Now consider the point $x^* - \delta r$. Observe that

$$r' \cdot x^* - \delta r \cdot r' > r' \cdot x^* - \delta, \ \forall r' \in \omega(a, \beta)$$

because $r \cdot r' < 1$. Hence, for a sufficiently small δ,

$$r' \cdot x^* - \delta > r' \cdot \frac{a + b(r')}{2}, \ \forall r' \in \omega(a, \beta).$$

Thus, $x^* - \delta r$ is in $GS(a, \beta)$. Notice that for direction r,

$$r \cdot (x^* - \delta r) = r \cdot x^* - \delta$$

because r has unit length. Clearly, $r \cdot x^* - \delta < r \cdot x^*$, which contradicts the assumption that x^* is on a supporting hyperplane of $GS(a, \beta)$.

Hence, $S(r) = H(a, b(r))$. Because $M(r)$ is a hyperplane with the same normal vector as $H(a, b(r))$, if $M(r)$ supports $GS(a, \beta)$, it must be coincident with $H(a, b(r))$. If it is further from β than $H(a, b(r))$, then $M(r) \subset H(a, b(r))^{++}$. Because $a \succeq b$ for every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^+$ by Lemma 4, it is therefore necessary for $M(r)$ to be weakly further from β than $S(r)$. □

Proof of Lemma 6. Consider a point $x \in \text{bd}(GS(a, \beta))$ which is supported by more than one hyperplane. Because a convex set is the intersection of its supporting hyperplanes, we know that the supporting hyperplanes at x are hyperplanes of equidistance. Consider two of the supporting hyperplanes, chosen arbitrarily if there are more than two. Because x is on both hyperplanes, it must be equidistant to two points in β, say b_1 and b_2, and that distance must equal $||a - x||$ by Proposition 1. By the convexity of β, there is a unique closest point to x in β. If $||a - x||$ is the minimal distance from x to a point in β, this contradicts uniqueness. If it is not the minimal distance, then there exists another point b_3 that is strictly closer to x, and therefore $||b_3 - x|| < ||a - x||$. But then $x \notin GS(a, \beta)$, by Proposition 1. Therefore, $x \in GS(a, \beta)$ cannot have multiple supporting hyperplanes when β is convex. □

Proof of Theorem 2. Consider the solution x^* to

$$\min r \cdot x \text{ subject to } x \in GS(a, \beta)$$

for an arbitrarily chosen $r \in \omega(a, \beta)$. By Lemma 6, The solution x^* must lie on a unique $S(r)$, because $r \cdot x \geq r \cdot x^*$ for all $x \in GS(a, \beta)$. By Lemma 5, it is therefore necessary that $M(r)$ be weakly further from β than $S(r)$. Because this is true for every $r \in \omega(a, \beta)$, it is therefore necessary that $M(r)$ be weakly further from β than $S(r)$ for every $r \in \omega(a, \beta)$.

By Theorem 1, it is sufficient that $M(r)$ be weakly further from β than $S(r)$. □

Proof of Theorem 3. First, if $a \notin \beta$, then $a \neq b(r)$ for any vulnerable direction $r \in \omega(a, \beta)$, because $b(r) \in \beta$.

If $-r \in \omega(a, \beta)$, then it is also the case that $b(r) \neq b(-r)$. Suppose, contrary to the claim, that $b(r) = b(-r)$. Then by definition, there exists γ_1 and γ_2 such that $a - \gamma_1 r = a + \gamma_2 r$. Then $-\gamma_1 r = \gamma_2 r$. The right hand side is nonnegative, so γ_1 must be 0. But then $b(r) = a$, contradicting $a \notin \beta$.

In order for a to be majority preferred to $b(r)$ and $b(-r)$, by Lemma 4 it must be true that $M(r) \subset H(a, b(r))^+$ and $M(-r) \subset H(a, b(-r))^+$. By Lemma 2, $M(r) = M(-r)$, so the condition can be stated as

$$r \cdot \frac{a + b(r)}{2} \leq r \cdot m(r) \leq r \cdot \frac{a + b(-r)}{2}.$$

Since $a \in H(a, b(r))^+$ and $a \in H(a, b(-r))^+$ by definition, a median hyperplane with direction r passing through a must be in $H(a, b(r))^+ \cap H(a, b(-r))^+$.

We now show that $H(a, b(r))^+ \cap H(a, b(-r))^+$ always has a nonempty interior. Consider an open ball of radius $\epsilon > 0$ centered at a. Because $a \neq b(r)$, we know that $b(r) = a - \gamma_1 r$ for $\gamma_1 > 0$. Similarly, as $a \neq b(-r)$, we know that $b(-r) = a + \gamma_2 r$ for $\gamma_2 > 0$. Choose $\epsilon \leq \min\{\frac{\gamma_1}{2}, \frac{\gamma_2}{2}\}$. We now show that $r \cdot \frac{a+b(r)}{2} \leq r \cdot x$. We have the following inequalities

$$\begin{aligned} r \cdot \frac{a + a - \gamma_1 r}{2} &\leq r \cdot x \\ r \cdot a - \frac{\gamma_1}{2} &\leq r \cdot x \\ r \cdot (a - x) &\leq \frac{\gamma_1}{2} \end{aligned}$$

The first follows from substitution for $b(r)$. The second follows from the fact that $r \cdot r = 1$, because r has unit length. Then because $r \cdot (a - x) < ||a - x||$, the inequality becomes

$$||a - x|| \leq \frac{\gamma_1}{2}.$$

Further, because $||a - x|| < \epsilon$, we have that

$$\epsilon \leq \frac{\gamma_1}{2},$$

which is true by assumption. The argument that $r \cdot x \leq r \cdot \frac{a+b(-r)}{2}$ is symmetric. Because x was chosen arbitrarily, we have shown that an open ball centered at a of positive radius is contained in $H(a, b(r))^+ \cap H(a, b(-r))^+$, and therefore a is an interior point, and thus, the interior is nonempty. □

Proof of Lemma 7. Assume, contrary to our claim, that there exists $r \in \omega(a, \beta)$ such that $-r \in \omega(a, \beta)$. Therefore, there exist distinct $b(r), b(-r) \in \beta$. But then a lies on a line connecting $b(r)$ and $b(-r)$. Because β is convex, $a \in \beta$, contradicting the assumption that $a \notin \beta$. □

Proof of Theorem 4. Immediate from Lemma 7 and Lemma 4. □

Proof of Lemma 8. Suppose $Y(c, \tau) \subset GS(a, \beta)$, but there exists an $r \in \omega(a, \beta)$ such that $M(r) \not\subset H(a, b(r))^+$ By definition of $Y(c, \tau)$, there must exist at least one $y \in Y(c, \tau)$ such that $y \in M(r)$. But then $y \notin H(a, b(r))^+$, contradicting $Y(c, \tau) \subset GS(a, \beta)$. Thus, $M(r) \subset H(a, b(r))^+, \forall r \in \omega(a, \beta)$, and by Lemma 4, a is a DCW. By a parallel argument, if $Y(c, \tau) \subset \text{int}GS(a, \beta)$ then $M(r) \subset H(a, b(r))^{++}, \forall r \in \omega(a, \beta)$, and by Lemma 4, a is an SDCW. □

Proof of Lemma 9. First, we show that f is positive. For every $x \in \Gamma$, $f(x)' > 0$ because $f(x) > \epsilon$. For every $x \in \Gamma'$, $f(x)' > 0$ because $\epsilon > 0$ and $f(x) > 0$. For every $x \notin \Gamma \cup \Gamma'$, $f(x)' > 0$ because $f(x) > 0$. Therefore, $f(x)' > 0,\ \forall x \in \mathbb{R}^n$.

Second, we show that $\int_{\mathbb{R}^n} f(x)' = 1$, using the following equivalences:

$$\begin{aligned}
\int_{\mathbb{R}^n} f(x)' &= \int_{\Gamma} f(x)' + \int_{\Gamma'} f(x)' + \int_{(\Gamma \cup \Gamma')^c} f(x)' \\
&= \int_{\Gamma} f(x) - \epsilon + \int_{\Gamma'} f(x) + \epsilon + \int_{(\Gamma \cup \Gamma')^c} f(x) \\
&= \int_{\mathbb{R}^n} f(x) - \int_{\Gamma} \epsilon + \int_{\Gamma'} \epsilon \\
&= \int_{\mathbb{R}^n} f(x) \\
&= 1
\end{aligned}$$

The first equality holds by decomposing the region of integration. The second holds by substituting the definition of $f(x)'$ on each region. The third holds by the linearity of integration. The fourth comes from computation of the value of the second two integrals, and the last from the fact that f is a probability distribution. Therefore, f' integrates to one. □

Proof of Lemma 10. For any d and d' such that $d' > d$, we have that

$$\int_{\{M(r) - d'r\}^+} f(x)' = \int_{\{M(r) - d'r\}^+ \setminus \{M(r) - dr\}^+} f(x)' + \int_{\{M(r) - dr\}^+} f(x)'.$$

By Lemma 9, f' is strictly positive. Because the first term on the right hand side is the integral of a strictly positive function on a region of positive measure, it is strictly positive. Hence,

$$\int_{\{M(r)-d'r\}^+} f(x)' > \int_{\{M(r)-dr\}^+} f(x)'.$$

□

Proof of Lemma 11. If $\int_{M(r)^+} f(x)' < 1/2$, then $M(r)' = M(r) - dr$ for some $d > 0$, because the integral is strictly increasing in d by Lemma 10. Hence, $M(r)' \not\subset M(r)^+$.

If $M(r)' \not\subset M(r)^+$, then $M(r)' = M(r) - dr$ for some $d > 0$. By definition, $\int_{\{M(r)-dr\}^+} f(x)' = \frac{1}{2}$. Setting $d = 0$ implies $\int_{M(r)^+} f(x)' < \frac{1}{2} = \int_{M(r)^+} f(x)$, because $\int_{\{M(r)-dr\}^+} f(x)'$ is strictly increasing in d by Lemma 10. □

Proof of Proposition 3. We show that for any direction r for which $r^* \cdot r \geq 0$, $M(r)' \subset M(r)^+$. First, by decomposing the region of integration, we see that

$$\int_{M(r)^+} f(x)' = \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\Gamma' \cap M(r)^+} \epsilon.$$

Consider r such that $r \cdot r^* \geq 0$ and pick any $x \in \Gamma \cap M(r)^+$. By definition, $x + \delta r^* \in \Gamma'$, and further

$$r \cdot (x + \delta r^*) \geq r \cdot x \geq r \cdot m(r).$$

Hence,

$$x \in \Gamma \cap M(r)^+ \implies x + \delta r^* \in \Gamma' \cap M(r)^+.$$

Therefore, the measure of $\Gamma \cap M(r)^+$ is no larger than the measure of $\Gamma' \cap M(r)^+$. Thus, $\int_{\Gamma \cap M(r)^+} \epsilon \leq \int_{\Gamma' \cap M(r)^+} \epsilon$. Hence, we can now see that

$$\int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\Gamma' \cap M(r)^+} \epsilon \geq \int_{M(r)^+} f(x).$$

By Lemma 11, $M(r)' \subset M(r)^+$, and therefore $M(r)'$ is either equal to $M(r)$ or further from β.

We now show that the distance between median hyperplanes is bounded by δ if r is such that $r^* \cdot r < 0$. Consider the integral of the new density function on the positive halfspace of $M(r)$:

$$\int_{M(r)^+} f(x)' = \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\Gamma' \cap M(r)^+} \epsilon.$$

Consider $x \in \Gamma' \cap M(r)^+$. By definition, $x - \delta r^* \in \Gamma$, and further,

$$r \cdot (x - \delta r^*) = r \cdot x - \delta r \cdot r^* > r \cdot x$$

because $r \cdot r^* < 0$. Because $r \cdot x \geq r \cdot m(r)$, it follows that $x - \delta r^* \in M(r)^+$. Thus, $x \in \Gamma' \cap M(r)^+ \implies x - \delta r^* \in \Gamma \cap M(r)$. Consequently, the measure of $\Gamma \cap M(r)^+$ is greater than or equal to the measure of $\Gamma' \cap M(r)^+$, and thus, $\int_{\Gamma \cap M(r)^+} \epsilon \geq \int_{\Gamma' \cap M(r)^+} \epsilon$.

If $\int_{\Gamma \cap M(r)^+} \epsilon = \int_{\Gamma' \cap M(r)^+} \epsilon$, then $\int_{M(r)^+} f(x)' = \int_{M(r)^+} f(x)$ and $M(r)$ is the median hyperplane when the distribution is f'. Hence $d = 0$.

If $\int_{\Gamma \cap M(r)^+} \epsilon > \int_{\Gamma' \cap M(r)^+} \epsilon$, then $\int_{M(r)^+} f(x)' < \frac{1}{2}$, and thus $M(r)' = M(r) - dr$ for some $d > 0$ by Lemma 11. We now show that $d \leq \delta$.

Let $\mathbb{1}\{x \in \Gamma\}$ be an indicator function equal to one if and only if $x \in \Gamma$. Then

$$\int_{M(r)^+ - dr} f(x) - \int_{\Gamma \cap M(r)^+ - dr} \epsilon = \int_{M(r)^+ - dr} f(x) - \epsilon \mathbb{1}\{x \in \Gamma\}.$$

The function $f(x) - \epsilon \mathbb{1}\{x \in \Gamma\}$ is strictly positive because $f(x) > \epsilon$. By a parallel argument to Lemma 10, $\int_{M(r)^+ - dr} f(x) - \epsilon \mathbb{1}\{x \in \Gamma\}$ is strictly increasing in d. Therefore,

$$\int_{\{M(r)-\delta r\}^+} f(x) - \int_{\Gamma \cap \{M(r)-\delta r\}^+} \epsilon > \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon.$$

We now show that $x \in \Gamma \cap M(r)^+ \implies x + dr^* \in \Gamma' \cap \{M(r) - \delta r\}^+$. By definition, $x + \delta r^* \in \Gamma$. What must be shown is that $x + \delta r^* \in \{M(r) - \delta r\}^+$. Observe that

$$r \cdot (x + \delta r^*) \geq r \cdot m(r) - \delta \iff r \cdot x + \delta r \cdot r^* \geq r \cdot m(r) - \delta.$$

The condition on the right hand side must be met, because $r \cdot x \geq r \cdot m(r)$ and $r \cdot r^* \geq -1$. Thus, if $x \in \Gamma \cap M(r)^+$ then $x + dr^* \in \Gamma' \cap \{M(r) - \delta r\}^+$, and the measure of $\Gamma \cap M(r)^+$ is no greater than the measure of $\Gamma' \cap \{M(r) - \delta r\}^+$.

Now we consider the integral of f' over $\{M(r) - \delta r\}^+$:

$$\int_{\{M(r)-\delta r\}^+} f(x)' = \int_{\{M(r)-\delta r\}^+} f(x) - \int_{\Gamma \cap \{M(r)-\delta r\}^+} \epsilon + \int_{\Gamma' \cap \{M(r)-\delta r\}^+} \epsilon.$$

Because

$$\int_{\{M(r)-\delta r\}^+} f(x) - \int_{\Gamma \cap \{M(r)-\delta r\}^+} \epsilon > \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon,$$

it must be true that

$$\int_{\{M(r)-\delta r\}^+} f(x)' > \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\Gamma' \cap \{M(r)-\delta r\}^+} \epsilon.$$

Because the measure of $\Gamma \cap M(r)^+$ is no greater than the measure of $\Gamma' \cap \{M(r) - \delta r\}^+$,

$$\int_{\Gamma \cap M(r)^+} \epsilon \leq \int_{\Gamma' \cap \{M(r)-\delta r\}^+} \epsilon.$$

Thus,

$$\int_{\{M(r)-\delta r\}^+} f(x)' > \int_{M(r)^+} f(x) = \frac{1}{2}.$$

Since the left hand side is strictly increasing in d, this implies that the d required to obtain a median hyperplane for the distribution f' is no greater than δ. $\square$

Proof of Theorem 6. Because a is a DCW when ideal points are distributed according to f, we know that $M(r) \subset H(a, b(r))^+$ for every $r \in \omega(a, \beta)$ by Proposition 2. By Proposition 3, when ideal points are distributed according to f', $M(r)' \subset M(r)^+$, and hence, $M(r)' \subset H(a, b(r))^+$ for every $r \in \omega(a, \beta)$. Thus by Proposition 2, a is a DCW. □

Proof of Theorem 7. Because a is an SDCW when ideal points have density $f(x)$, we know that $M(r) \subset H(a, b(r))^{++}$ for every $r \in \omega(a, \beta)$ by Proposition 2. This implies that $r \cdot m(r) > r \cdot \frac{a+b(r)}{2}$ for every $r \in \omega(a, \beta)$. By Proposition 3, $M(r)' = M(r) - dr$ for $d < \delta$. By choosing δ to be sufficiently small, $r \cdot m(r)' > r \cdot m(r) - \delta > r \cdot \frac{a+b(r)}{2}$ for every $r \in \omega(a, \beta)$. Therefore, $M(r)' \in H(a, b(r))^{++}$ for every $r \in \omega(a, \beta)$, so by Proposition 2, a is an SDCW when ideal points are distributed with density f. □

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