Two Candidate Competition on Differentiated Policy Sets

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Abstract

In the classical spatial model of two candidate competition, an equilibrium exists only if the distribution of voter ideal points is such that every median hyperplane passes through a single policy. The necessity of this condition crucially depends upon both candidates being able to propose any policy in a Euclidean space. We assume that each candidate is affiliated with a party which restricts the policies that its candidate can propose and that voters have Euclidean spatial preferences. We show that if the candidates can only make proposals from disjoint sets of policies, then an equilibrium exists if each median hyperplane passes through a region with a nonempty interior that contains the equilibrium policy. An equilibrium, if it exists, is generically robust to perturbations of the voters' ideal points.

1 Introduction

In the classical spatial model of elections, two office motivated candidates can each propose policies in a Euclidean space. In a pure strategy equilibrium, each candidate must propose a Condorcet winner, a policy that is majority preferred to all others. However, a Condorcet Winner exists only under highly restrictive circumstances. Specifically, if there is an odd number of voters, the utility gradients of the voters must be pairwise symmetric at the equilibrium policy. That is, for every individual who prefers to move the policy in a given direction from the equilibrium, there is another voter who prefers to move the policy in the opposite direction (Plott, 1967). Similar pairwise symmetry requirements are also necessary if both candidates choose from a common convex set of permissible policies (Matthews, 1980). When the number of voters is even, similar necessary conditions apply; however, these conditions need not be sufficient (Duggan, 2018). If the distribution of voters does not have any policy at which the utility gradients of the voters are pairwise symmetric, a Condorcet Winner does not exist. In this case, any policy can be chosen by a sequence of votes even if it is Pareto dominated (McKelvey, 1976). The results of Plott (1967), Matthews (1980), and Duggan (2018) imply that an equilibrium policy for either candidate exists only in knife-edge cases.[1]

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[1] These issues have also been considered by Sloss (1973), Cohen and Matthews (1980), McKelvey et al. (1980), Schofield (1983), and Chung and Duggan (2018). All of these papers assume that the candidates choose from the same set of permissible policies.

In this paper, we analyze whether an equilibrium exists more generally when voters have Euclidean spatial preferences if each candidate may only propose policies from a subset of the policy space and the feasible subsets of the two candidates are disjoint. We investigate the existence and properties of a *Restricted Condorcet Winner*, which is a policy for one candidate that is majority preferred to every policy that the other candidate can propose.

Our main results are as follows. Given a policy a and a closed set of possible opponent policies, a voter (who is characterized by his ideal point) is a *guaranteed supporter* of a if for every policy the opponent can propose, the voter prefers a to it. The guaranteed supporter set is the set of all such voter ideal points. A policy a is a Restricted Condorcet Winner if, for every direction from a in which an opponent may propose a policy, the median hyperplane for that direction is either tangent to the guaranteed supporter set or further from the opponent's policy set.[2] When the set of permissible policies for the opponent is also convex, then the number of directions that must be considered is at most half the number in the classic spatial model, and the guaranteed supporter condition is also necessary. In contrast, the policy a is an equilibrium outcome in the classical spatial model if and only if every median hyperplane passes through a. Consequently, an equilibrium policy exists in more general circumstances in our model than in the classical model. Further, we find that equilibria are robust to perturbations of the voters' ideal points in two ways. First, there are some directions in which voters' ideal points could move by an arbitrary distance while preserving a as an equilibrium. Second, if a is an equilibrium with a strict majority, then a perturbation of voters in any direction by a sufficiently small distance preserves a as an equilibrium.

The Plott conditions are necessary at an equilibrium policy in the classical spatial model because for any policy proposed by one candidate, the opponent can propose a policy in any direction and at any distance from it. The fact that infinitesimal deviations are possible places restrictions on the gradients of the voters' utility functions, and in conjunction with the fact that these deviations may be in any direction, it implies that these gradients must be pairwise symmetric. In the one-dimensional spatial model pioneered by Downs (1957) and Hotelling (1929), if voters have Euclidean preferences, the Median Voter Theorem states that the ideal point of a median voter is the Condorcet Winner (Black, 1958). At a median voter's ideal point, half of the remaining voters prefer to go left, and half of them prefer to go right. Hence, pairwise symmetry is ensured in the one-dimensional strategy space at the median ideal point. In higher dimensions, pairwise symmetry may not occur at any proposed policy because when voters have Euclidean preferences, pairwise symmetry is satisfied only at a total median of the distribution. Consequently, in order for an equilibrium to exist in the absence of a total median, it must be the case that Plott's assumptions do not apply. This occurs if there is restriction on the directions in which alternative policies may lie, their distance from the Condorcet Winner, or both. As noted above, Matthews (1980) considers a common convex set of permissible policies, which impose restrictions on the directions in which a policy can be changed. However, Matthews finds pairwise symmetry requirements similar to Plott (1967), albeit for a subset of the directions of deviation from the equilibrium policy.[3] Our model imposes a restriction on both the distance between and direction of competing policies. Because both candidates have disjoint policy sets, competing policies cannot be arbitrarily close to

[2] Note that the opponent must move a finite distance from a in order to find an permissible policy for herself.

[3] In related work, Shepsle and Weingast (1981) consider common restrictions on the alternatives that may compete which effectively restrict attention to a one dimensional set of alternatives at a time, in which case the median position on each issue is an equilibrium policy. However, these restrictions are absent in the kind of two candidate elections considered here as it is impossible for voters to vote on one dimension at a time.

one another, and competing policies cannot lie in any direction from one another. Hence, in the environment we consider, the assumptions underlying the Plott conditions are not met.

Political parties are a natural mechanism for constraining candidates to disjoint policy sets, as we illustrate in the following example. If a candidate chooses to run for office as a Republican, she may be unable to propose gun control of any kind and, instead, may be required to be at least slightly in favor of gun rights. On defense, she may be unable to propose a budget reduction. It may be that maintaining the current level of spending is the least pro-defense policy she is able to propose. A Democrat faces the opposite constraints; he may be required to propose at least some level of gun control, and at least marginal cuts to defense spending. The party structure restricts the distance between competing policies, as a Democrat cannot credibly propose a policy arbitrarily close to a Republican pledging to double the defense budget and overturn waiting periods for gun purchases. The party structure also restricts the directions in which competing policies may lie. The Democrat cannot credibly pledge a larger defense budget or weaker gun laws than the Republican. Consequently, the Republican's policy can only ever compete against policies that involve less spending, stronger gun laws, or both. Even if it were the case that a majority of voters prefer more defense spending, the Democrat cannot propose such a policy.

In reality, parties may not have disjoint policy sets. What is important for our analysis is that parties do not have identical sets of permissible policies; they need not be disjoint. However, if an equilibrium policy is on the boundary of or inside of the opponent's set of permissible policies, then the situation is the same as the one considered by Matthews (1980) or by Plott (1967). The equilibrium conditions that must hold in such circumstances are already well understood, and so they are not considered further here. For this reason, we restrict attention to the case in which the candidates have disjoint sets of policies.

In recent years there has been significant interest in models of differentiated candidates in which candidates exogenously differ on some dimensions, but can compete on others. Because there is a dimension in which the candidates differ, their policy sets are disjoint, as is the case here.[4] However, because they can freely choose the values in all but one dimension, these policy sets have a special structure. In contrast, we only require that the policy sets are closed and possibly convex in addition to being disjoint. To the best of our knowledge, there is no paper considering the existence of a majority preferred policy in an environment with general candidate-specific spatial constraints.

In one strand of the literature on differentiated candidate, the policies in the fixed dimension cannot be credibly altered because of characteristics or past behavior of the candidates. The focus of these papers is on how the values of the policies in the fixed dimension affect the equilibrium choice of policies in the flexible dimension. In the case of Krasa and Polborn (2010, 2014) and Dziubiński and Roy (2011), candidates have fixed positions in one dimension and compete in a second. While the policy space is two dimensional, the constraint on one issue reduces the strategy space for each candidate to one dimension, and so a pure-strategy equilibrium exists in more general circumstances than in the classic spatial model. In the case of Soubeyran (2009), Krasa and Polborn (2012), and Matakos and Xefteris (2017), the fixed dimension is non-spatial and the flexible dimensions are spatial. These papers find that a pure strategy equilibrium exists in a multidimensional strategy space as a result of this non-spatial differentiation.

[4]In a three voter model, Asay (2008) considers linear restrictions on the candidate's policy sets due to ideological commitments, but does not require them to be disjoint. For the most part, he is interested in determining a policy for a candidate that minimizes the size of the set of policies that can defeat it.

While Xefteris (2017) does not consider why the candidates differ on the fixed dimension, he also assumes that candidates are flexible on $n-1$ issues and fixed on the other one. He finds that a unique Nash equilibrium exists when candidates maximize their vote share (rather than seeking a majority preferred policy) and the candidates are sufficiently differentiated on the fixed issue.

The rest of this paper is organized as follows. In Section 2, we introduce the formal model and the solution concept. In Section 3, we define the guaranteed supporter set and median hyperplanes and in Section 4 we show how they can be used to characterize the conditions for the existence of an equilibrium. In Section 5, we show that these conditions are weaker than those in the classic spatial model, and that an equilibrium therefore exists in more general circumstances. In Section 6, we consider perturbations of the distribution of voters and establish some comparative static results. In Section 7, we provide some concluding remarks. Proofs of all results are in the appendix.

2 Restricted Condorcet Winners

In this section, we introduce the formal model and our solution concept, the Restricted Condorcet Winner. We then provide intuition for why a Restricted Condorcet Winner may be expected to exist more generally than a standard Condorcet Winner.

2.1 Preliminaries

There are n dimensions of policy, so that any particular policy is a vector $x \in \mathbb{R}^n$. The set of all possible policies is $\mathbb{R}^n$. Voters have utility given by $U(x, \theta) = -||x - \theta||^2$, where $\theta \in \mathbb{R}^n$ is the ideal point of the voter. The density of voter ideal points, $f : \mathbb{R}^n \mapsto \mathbb{R}_{++}$, is continuous with full support. Voters are identified by their ideal points, so we often speak of the measure of voters on a set, rather than the measure of voter ideal points on the set. Voters vote sincerely for their most preferred alternatives.

There are two office motivated candidates, A and B. Candidate A chooses a policy $a \in \alpha$ and candidate B chooses a policy $b \in \beta$. *As maintained assumptions, it is assumed that α and β are disjoint and that both α and β are (i) non-degenerate (i.e., each of these sets contains at least two policies) and (ii) closed.*[5] Closedness ensures that for any policy of one candidate, there is a closest policy to it among the policies available to the other candidate. Additional assumptions on α and β, such as convexity, are added as they are used in Sections 4 and 5.

Example. To illustrate policy sets satisfying our assumptions, consider the example in the introduction. Suppose that candidates are constrained by their respective parties to propose policies opposite one another on two issues: gun rights x and defense spending y. We assume that candidate A, the Republican, is restricted to policies that are no less than ε on both issues, while candidate B, the Democrat, is restricted to policies no greater than $-\varepsilon$ on each of the two issues. Therefore, $\alpha \equiv \{(x, y) \in \mathbb{R}^2 | x \geq \varepsilon, y \geq \varepsilon\}$ and $\beta \equiv \{(x, y) \in \mathbb{R}^2 | x \leq -\varepsilon, y \leq -\varepsilon\}$. These policy sets are shown in Figure 1.

Consider a hyperplane H in $\mathbb{R}^n$ with normal vector in direction r containing point p. Associated with H are the following halfspaces:

$$H^+ = \{x \in \mathbb{R}^n \mid x \cdot r \geq p \cdot r\},$$

[5] These assumptions are not stated explicitly in our results.

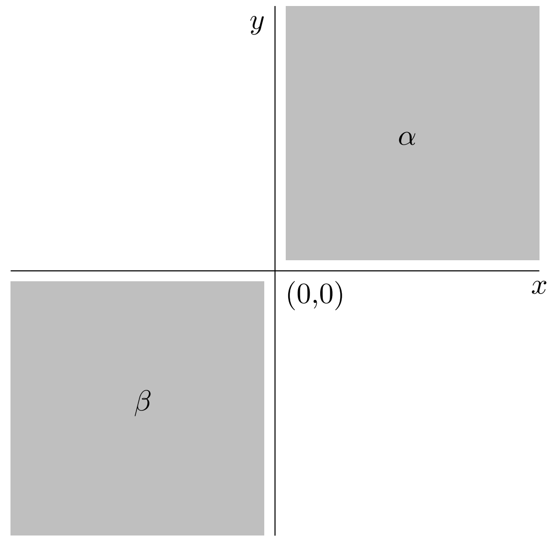

Figure 1: Policy sets for A and B in two party competition.

$$H^{++} = \{x \in \mathbb{R}^n \mid x \cdot r > p \cdot r\},$$

and

$$H^{-} = \{x \in \mathbb{R}^n \mid x \cdot r \leq p \cdot r\}.$$

Definition 1. The set

$$H(a,b) = \{x \in \mathbb{R}^n \mid x \cdot r_{ab} = \frac{a+b}{2} \cdot r_{ab}\}$$

is the *hyperplane of equidistance for a and b.*

The set $H(a,b)$ is a hyperplane through the midpoint of the line connecting a and b with normal vector $r_{ab} = \frac{a-b}{||a-b||}$. Note that by construction, r_{ab} is a unit vector, and points from the midpoint toward a. Denote the halfspace containing a by $H(a,b)^+$, which is the set of points no further from a than b. The interior of the halfspace is denoted $H(a,b)^{++}$, which is the set of points strictly closer to a than b. An example is depicted in Figure 2.

Note that for any point x, $||a-x|| \leq ||b-x|| \iff x \in H(a,b)^+$ and $||a-x|| < ||b-x|| \iff x \in H(a,b)^{++}$. A voter with ideal point θ therefore prefers a over b if and only if $\theta \in H(a,b)^{++}$.

Let $\succeq$ represent weak majority preference, and $\succ$ represent strict majority preference. Then

$$a \succeq b \iff \int_{H(a,b)^+} f(x) \geq 1/2$$

and

$$a \succ b \iff \int_{H(a,b)^+} f(x) > 1/2.$$

The set of indifferent voters is of measure zero in $\mathbb{R}^n$, so they are ignored.

We carry out our analysis from the perspective of candidate A without loss of generality. All of the results hold for candidate B with an appropriate replacement of a with b and α with β.

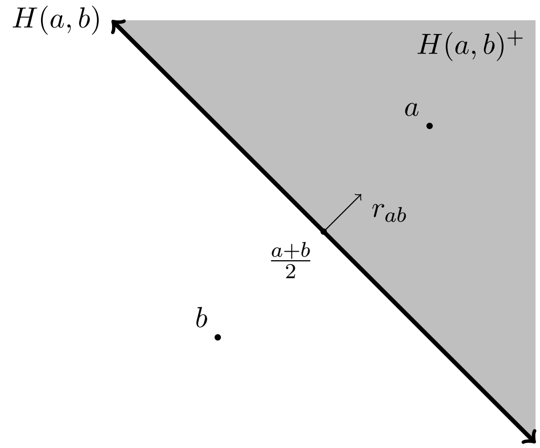

Figure 2: The hyperplane of equidistance and its associated halfspaces.

2.2 A Restricted Condorcet Winner

A Condorcet Winner is a policy that is weakly majority preferred to any other policy that can be proposed.

Definition 2. A policy a is a *Condorcet Winner (CW)* if and only if $a \succeq b, \ \forall b \in \mathbb{R}^n$.

The Condorcet Winner is an inappropriate solution concept for situations in which two candidates have disjoint sets of policies they can propose, as a Condorcet Winner must be majority preferred to every other alternative in the policy space. We seek a policy $a \in \alpha$ that is majority preferred to every policy $b \in \beta$, rather than every policy $b \in \mathbb{R}^n$. We refer to such a policy as a *Restricted Condorcet Winner.*

Definition 3. A policy a is a *Restricted Condorcet Winner on β (RCW)* if and only if $a \succeq b, \ \forall b \in \beta$. If $a \succ b, \ \forall b \in \beta$, then a is a *Strict Restricted Condorcet Winner on β (SRCW).*

The following example of four alternatives and three voters illustrates the distinction between a CW and an RCW. Each individual's strict preferences over policies w, x, y, and z are listed in descending order. Suppose that w and x are the policies candidate A may propose, and that y and z are the policies candidate B may propose.

Voter J	Voter K	Voter L
x	y	w
y	z	x
w	w	y
z	x	z

Note that there is a majority preference cycle; $x \succeq y$, $y \succeq w$, and $w \succeq x$. This cycle precludes x, w, or y from being Condorcet Winners. Further, $y \succeq z$, so z cannot be one either, and, hence, there is no Condorcet Winner. However, despite being an element of the cycle, policy x is an RCW because policy x is majority preferred to both y and z, the only policies B can propose. While policy x would lose to w, because policies w and x are both in A's policy set, a vote between w and x cannot occur. The partition of the policies between A and B prevents the preference cycle from being realized as a sequence of votes, so an RCW exists.

In the classical spatial model with Euclidean preferences, if a Condorcet Winner does not exist, then there is a majority preference cycle which encompasses the entire policy space (Cohen, 1979). That is, a majority preference cycle encompassing the entire policy space is a necessary condition for the nonexistence of a Condorcet Winner. Restricting the sets of alternatives for each candidate may stabilize the outcome of majority rule because it prevents some cycles in the majority preferences of the voters from being possible sequences of votes. Specifically, because one set of policies can only be proposed by A and the other by B, any preference cycle contained entirely in α or β cannot be a sequence of votes. For example, suppose that $a_1 \succeq a_2$, $a_2 \succeq a_3$, and $a_3 \succeq a_1$, with $a_1, a_2, a_3 \in \alpha$. While this cycle exists in the voters' preferences, it cannot exist as a sequence of votes because B cannot propose a_2. Further, any cycle which contains two consecutive elements of α or β cannot be a sequence of votes either. If the only policy that is majority preferred to a_j is a_k, then a_j is an RCW because a_k cannot be proposed by B. The only cycles in votes that remain when candidates have disjoint policy sets are those which alternate between α and β for each element of the sequence.

3 Guaranteed Supporter Sets and Median Hyperplanes

In this section, we define the guaranteed supporter set and the median hyperplanes for a distribution of voters. The guaranteed supporter set considers the set of voters who prefer a policy a to every policy the opponent can propose. We show that this set can be characterized either as an intersection of halfspaces of equidistance, or as a set of ideal points for which a policy a is closer than the closest policy $b \in \beta$.

In order for a to defeat every policy $b \in \beta$, for each b there has to be a way to find a group of at least half of the voters who prefer a to b. A starting point to identifying when this is possible is to identify the group of people who support a over every $b \in \beta$; that is, a group of guaranteed supporters. We show that this group can be identified by considering only the hyperplanes of equidistance for the policies closest to a in β.

To determine who is a guaranteed supporter of a, we first consider in what directions from a policies $b \in \beta$ may lie. Because these are the directions a may be challenged in, we say they are *vulnerable.*

Definition 4. A direction r is a *vulnerable direction at* a if and only if $\exists b \in \beta$ *s.t.* $b = a - \gamma r$, for some $\gamma > 0$.

Recall that directions are normalized to have a length of one. For any $b \in \beta$, there is a $\gamma > 0$ such that $b = a - \gamma r$ for some r, namely $r = \frac{a-b}{||a-b||}$ and $\gamma = ||a - b||$. The minus sign accounts for the fact that r is pointing from b toward a. The collection of directions required to reach every policy $b \in \beta$ is the *set of vulnerable directions*, $\omega(a, \beta)$.

We next consider how close a policy $b \in \beta$ can be relative to a, in each vulnerable direction. The set

$$\beta(r) \equiv \{b \in \beta | b = a - \gamma r \text{ for some } \gamma > 0\}$$

is the set of all policies in β that lie in direction r from a. Let $b(r)$ represent the closest policy

$b \in \beta$ to a in direction r, or formally,

$$b(r) \equiv \arg\min_{b \in \beta(r)} ||a - b||$$

The closedness of β ensures that $b(r)$ is defined. See Figure 3.

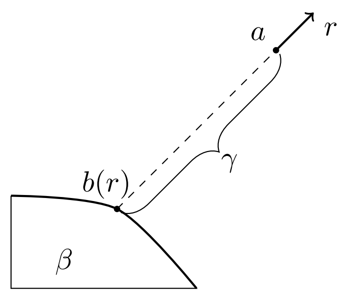

Figure 3: The closest policy in β to a in direction r, $b(r)$.

For each direction $r \in \omega(a, \beta)$, we now show that if θ is weakly closer to a than $b(r)$, then it is strictly closer to a than every $b \neq b(r)$ in $\beta(r)$. Thus, if a is preferred to $b(r)$, then a is strictly preferred to any policy b further from a in direction r.

Lemma 1. For any ideal point θ, for any policy $a \in \alpha$, any vulnerable direction r at a, and any $b \in \beta(r)$ for which $b \neq b(r)$, $||a - \theta|| \leq ||b(r) - \theta||$ implies $||a - \theta|| < ||b - \theta||$.

Consider a voter with ideal point θ and a policy a such that θ is not closer to b than a for any $b \in \beta$. Then for each direction $r \in \omega(a, \beta)$, θ must be closer to a than $b(r)$, or else that voter would prefer $b(r)$. By Lemma 1, if a is preferred to $b(r)$, then a is also preferred to any policy b further from a in direction r. Thus, a is weakly preferred to every b in direction r by a voter with ideal point θ if and only if $\theta \in H(a, b(r))^+$. In order for this to be true for every policy $b \in \beta$, it needs to be true that $\theta \in H(a, b(r))^+, \ \forall r \in \omega(a, \beta)$. Therefore, a voter prefers a to b for every $b \in \beta$ if and only if $\theta \in \bigcap_{r \in \omega(a,\beta)} H(a, b(r))^+$.

Definition 5. The *guaranteed supporter set* (GS) for (a, β) is

$$GS(a, \beta) \equiv \bigcap_{r \in \omega(a,\beta)} H(a, b(r))^+.$$

Note that the set $GS(a, \beta)$ is a closed convex set, as it is the intersection of closed halfspaces.

Proposition 1. For any policy $a \in \alpha$, an ideal point θ satisfies:

(i) $||a - \theta||^2 \leq ||b - \theta||^2, \ \forall b \in \beta$, if and only if $\theta \in GS(a, \beta)$.

(ii) $||a - \theta||^2 < ||b - \theta||^2, \ \forall b \in \beta$, if and only if $\theta \in \text{int}GS(a, \beta)$.

(iii) $||a - \theta||^2 \leq ||b - \theta||^2, \ \forall b \in \beta$ and $||a - \theta||^2 = ||b(r) - \theta||^2$ for some $r \in \omega(a, \beta)$ if and only if $\theta \in \text{bd}(GS(a, \beta))$.

Proposition 1 allows us to consider $GS(a, \beta)$ either as the intersection of halfspaces or as the set of ideal points that are closer to a than any $b \in \beta$. Both ways of defining this set are useful.

Example (Continued). To illustrate the guaranteed supporter set, we return to our running example. To derive the guaranteed supporter set for a given policy $a \in \alpha$, we use the characterization in Proposition 1 and look for voters who are weakly closer to a than the closest policy to them in β. Let θ_x be the x coordinate of θ and let θ_y be its y coordinate. Let $b^*(\theta)$ be the closest $b \in \beta$ to θ. Consider first a voter with $\theta_x \geq -\varepsilon$ and $\theta_y \geq -\varepsilon$. For such a voter, $b^*(\theta) = (-\varepsilon, -\varepsilon)$. Therefore, this voter is closer to a than $b^*(\theta)$, and thereby a guaranteed supporter for a, if and only θ is above the hyperplane of equidistance for a and $(-\varepsilon, -\varepsilon)$. For a voter with $\theta_x \leq -\varepsilon$ and $\theta_y \geq -\varepsilon$, $b^* = (\theta_x, -\varepsilon)$. Therefore, the locus of equidistant points is a parabola with a as its focus and $y = -\varepsilon$ as its directrix. This voter is only closer to a than $b^*(\theta)$ if θ is on or above this parabola. For a voter with $\theta_x \geq -\varepsilon$ and $\theta_y \leq -\varepsilon$, $b^* = (-\varepsilon, \theta_y)$. Therefore the locus of equidistant points is a parabola with a as its focus and the $x = -\varepsilon$ as its directrix. Hence, this voter is closer to a than $b^*(\theta)$ if and only if θ is located on or above this parabola. This results in the $GS(a, \beta)$ that appears in Figure 4.

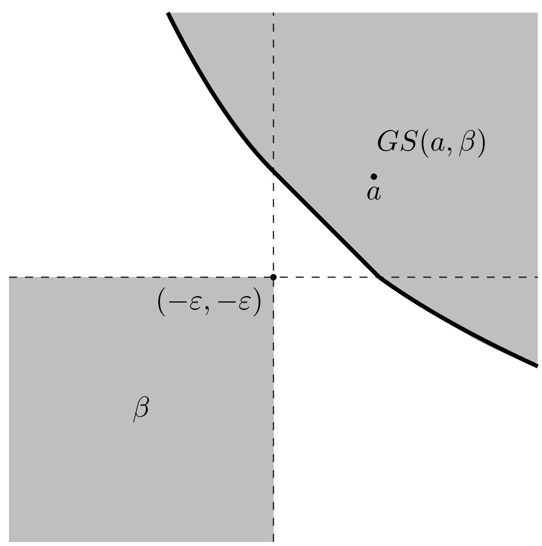

Figure 4: The GS set for a given policy $a \in \alpha$.

The *median hyperplane in direction* r, $M(r)$, has r as a normal vector and is situated so that half of the voters lie on one side of it and half lie on the other. This hyperplane is determined by the direction r and a point $m(r) \in \mathbb{R}^n$.

Definition 6. A *median hyperplane for direction* r is the set

$$M(r) = \{x \in \mathbb{R}^n | r \cdot x = r \cdot m(r)\},$$

where $m(r) \in \mathbb{R}^n$ is chosen so that

$$\int_{M(r)^+} f(x) \geq 1/2$$

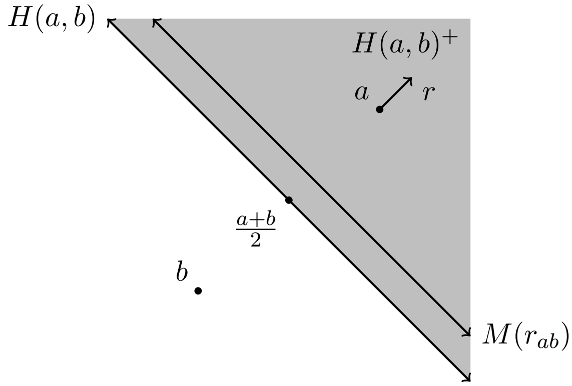

Figure 5: Policy a is strictly majority preferred to b.

and

$$\int_{M(r)^-} f(x) \geq 1/2.$$

Note that the point $m(r)$ used to define $M(r)$ is not unique. Any other point on this hyperplane could be used instead. Also note that the median hyperplane for direction r is coincident with the median hyperplane for direction $-r$; that is, $M(r) = M(-r)$.

4 Characterizing Restricted Condorcet Winners

In this section we describe the restrictions on the median hyperplanes that need to be satisfied for a policy to be a Restricted Condorcet Winner. We first consider the case in which β is only assumed to satisfy our maintained assumptions that β is nondegenerate and closed. We then consider the case in which β is also assumed to be convex.

The median hyperplanes characterize the majority preference over pairs of policies. For any two policies a and b, a is majority preferred to b if and only if $M(r_{ab}) \subset H(a,b)^+$. An example is depicted in Figure 5.

Lemma 2. For any $a, b \in \mathbb{R}^n, a \succeq b$ if and only if $M(r_{ab}) \subset H(a,b)^+$, with strict preference when $M(r_{ab}) \subset H(a,b)^{++}$.

Next, we extend Lemma 2 to show that only the closest policy in each direction must be considered.

Lemma 3. For any policy $a \in \alpha$ and any vulnerable direction r at a, a is majority preferred to every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^+$, with strict majority preference if and only if $M(r) \subset H(a, b(r))^{++}$.

To gain some geometric intuition for the conditions in Lemma 3, consider an initial situation in which $M(r) \subset H(a, b(r))^+$, as illustrated in Figure 6. Suppose that voters' preferences change so that $\bar{M}(r)$, the new median hyperplane for direction r, is a subset of $M(r)^{++}$. Initially, $r \cdot \frac{a+b(r)}{2} \leq r \cdot m(r)$. Because $\bar{M}(r) \subset M(r)^{++}$, it must be that $r \cdot \bar{m}(r) > r \cdot m(r)$. Hence, $r \cdot \frac{a+b(r)}{2} < r \cdot \bar{m}(r)$.

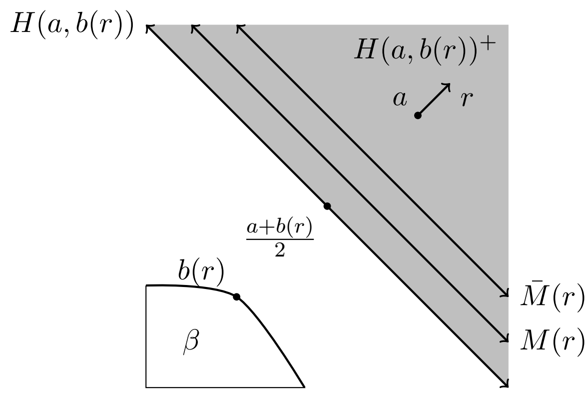

Figure 6: $\bar{M}(r)$ is strictly further from β in direction r than $M(r)$.

Any weak majority preference for a over $b(r)$ is now strict, and any strict majority preference is preserved. Intuitively, a voter with ideal point $\bar{m}(r)$ prefers to *move further from* β than one with ideal point $m(r)$.

Definition 7. For two hyperplanes H and $\bar{H}$ with the same normal vector $r \in \omega(a, \beta)$, $\bar{H}$ is *weakly further from* β than H if and only if $\bar{H} \subset H^+$ and is *strictly further from* β than H if and only if $\bar{H} \subset H^{++}$.

As illustrated in Figure 6, if $a \succeq b(r)$, then it must be that the median hyperplane $M(r)$ is weakly further from β than $H(a, b(r))$. We can now succinctly characterize what must be true of the median hyperplanes in order for a to be an RCW or an SRCW.

Proposition 2. A policy $a \in \alpha$ is an RCW if and only if $M(r)$ is weakly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$. A policy $a \in \alpha$ is an SRCW if and only if $M(r)$ is strictly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$.

Proposition 2 identifies the circumstances in which a policy a is an RCW in terms of the relationship between the median hyperplane in direction r and the hyperplane of equidistance in this direction, for every vulnerable direction r. We now show how this characterization of an RCW can be extended to characterize an RCW in terms of the guaranteed supporter set, $GS(a, \beta)$.

A natural conjecture is that a is an RCW if and only if all median hyperplanes are supporting hyperplanes of $GS(a, \beta)$ or are strictly further from β. After all, $GS(a, \beta)$ is the intersection of all of the halfspaces of equidistance. We show in Theorem 1 that this condition is sufficient. To formalize this argument, consider a vulnerable direction r. Then, because $GS(a, \beta)$ is convex, there is a unique supporting hyperplane with normal vector r, $S(r)$, to $GS(a, \beta)$. The median hyperplane is supporting to $GS(a, \beta)$ if and only if $M(r) = S(r)$. Consequently, all median hyperplanes are supporting hyperplanes of $GS(a, \beta)$ or are strictly further from β if $M(r)$ is weakly further from β than $S(r)$.

Theorem 1. If for every $r \in \omega(a, \beta)$, $M(r)$ is weakly further from β than $S(r)$, then a is an RCW. If for every $r \in \omega(a, \beta)$, $M(r)$ is strictly further from β than $S(r)$, then a is an SCDW.

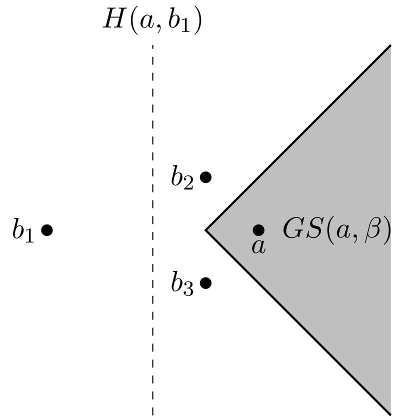

Figure 7: For $\beta = \{b_1, b_2, b_3\}$, the hyperplanes $H(a, b_2)$ and $H(a, b_3)$ support $GS(a, \beta)$, while $H(a, b_1)$ does not.

However, it may not be necessary for all median hyperplanes to support $GS(a, \beta)$. Difficulty arises when there are kinks in the boundary of $GS(a, \beta)$, as illustrated in Figure 7. A kink implies that some of the hyperplanes of equidistance may not support $GS(a, \beta)$ because the other constraints have effectively made them redundant. In the case of Figure 7, where $\beta = \{b_1, b_2, b_3\}$, b_1 is so far away from a that an ideal point that is closer to a than b_2 or b_3 is necessarily closer to a than b_1. At the kink on the boundary of $GS(a, \beta)$, $x \in H(a, b_2)$ and $x \in H(a, b_3)$, but $x \in H(a, b_1)^{++}$. As a consequence, a could be an RCW if the median line for r_{ab_1} is located between the kink point of $GS(a, \beta)$ and $H(a, b_1)$; it need not support $GS(a, \beta)$.

However, at any point on the boundary of $GS(a, \beta)$ that is uniquely supported by $S(r)$ for some $r \in \omega(a, \beta)$, it is in fact necessary that the median hyperplane $M(r)$ either coincides with $S(r)$ or is further from β in this direction than it.

Lemma 4. If $a \in \alpha$ is an RCW and there exists a hyperplane $S(r)$ that uniquely supports a point $x^* \in GS(a, \beta)$, then it is necessary that $M(r)$ is weakly further from β than $S(r)$.

A kink on the boundary of $GS(a, \beta)$ is necessarily equidistant to multiple policies in β. However, if β is closed and convex, we know that there must be a unique policy $b \in \beta$ that is closest to any policy $x \notin \beta$. As a consequence, there cannot be kinks on the boundary of $GS(a, \beta)$ if β is assumed to be convex in addition to our maintained assumption that it is closed.

Lemma 5. If β is convex, $GS(a, \beta)$ is uniquely supported at every point in $\text{bd}(GS(a, \beta))$.

It need not be the case that there is a solution to $\min_{x \in GS(a,\beta)} r \cdot x$ for every $r \in \omega(a, \beta)$, despite the fact that the value of the minimization is bounded from below by $r \cdot \frac{a+b(r)}{2}$ and $GS(a, \beta)$ is a closed set.[6] However, if there is and β is convex, then a necessary and sufficient condition for a to be an RCW is that all median hyperplanes are either supporting to $GS(a, \beta)$ or are further from β in direction r.

[6]For example, consider choosing $(x, y) \in \mathbb{R}^2_+$ to minimize y subject to $y \geq \frac{1}{x}$. This minimization is clearly bounded below by 0, and the constraint generates a closed admissible set, yet there is no solution.

Theorem 2. If β is convex and $\min_{x \in GS(a,\beta)} r \cdot x$ has a solution for every $r \in \omega(a, \beta)$, then a is an RCW if and only if $M(r)$ is weakly further from β than $S(r)$ for every $r \in \omega(a, \beta)$.

By the Weierstrass Theorem, the assumption in Theorem 2 that the minimization problem has a solution for every $r \in \omega(a, \beta)$ is satisfied if $GS(a, \beta)$ is compact.

Theorem 2 establishes that if the guaranteed supporter set satisfies a regularity condition, it can be used to characterize the conditions for a to be an RCW. Informally, the requirement that each $M(r)$ either supports $GS(a, \beta)$ or lies further from β is a requirement that the median voter for direction r can be chosen to be a guaranteed supporter of a, or to have a stronger preference for a over $b(r)$.

5 Comparison to the Classic Spatial Model

In the classical spatial model $\alpha = \beta = \mathbb{R}^n$. In this case, a policy is only an equilibrium if it is a total median. That is, it is a policy a such that every median hyperplane passes through it. We show that the necessary and sufficient conditions on the median hyperplanes for a given policy a to be an RCW in the model considered here are significantly weaker than in the classic spatial model. Rather than the requirement that every median hyperplane pass through a, we show that the median hyperplanes must only pass through a set with a nonempty interior. With further assumptions on β, at least half of the constraints are removed from consideration. We then consider the conditions for there to exist any policy that can be supported as an equilibrium, and find that these are weaker than those required in the classic spatial model.

5.1 Conditions for an Equilibrium

Theorem 2 provides the necessary and sufficient conditions on the median hyperplanes in order for a to be an RCW. In this subsection, we show that these conditions are in fact weaker than those in the classic spatial model.

To make this comparison, we first assume $\alpha = \beta = \mathbb{R}^n$ (thereby relaxing the assumption that $\alpha \cap \beta = \emptyset$), so that each candidate can propose any policy in $\mathbb{R}^n$, as in the classic spatial model. Under this assumption, all directions are vulnerable for any choice of a. Because Lemmas 2 and 3 are valid even if the policy sets are not disjoint, it is necessary in order for a to be an RCW that $M(r) \subset H(a, b(r))^+$. Further, because $M(r) = M(-r)$ and all directions are vulnerable, it must also be the case that $M(r) \subset H(a, b(-r))^+$. Therefore, if both r and $-r$ are vulnerable directions, a can be an RCW only if

$$M(r) \subset H(a, b(r))^+ \cap H(a, b(-r))^+,$$

or, equivalently, if

$$r \cdot \frac{a + b(r)}{2} \leq r \cdot m(r) \leq r \cdot \frac{a + b(-r)}{2}.$$

Because β is all of $\mathbb{R}^n$, $a \in \beta$. Consequently, $b(r) = b(-r) = a$ for any direction r. Therefore, $M(r)$ must satisfy

$$r \cdot \frac{a + a}{2} \leq r \cdot m(r) \leq r \cdot \frac{a + a}{2},$$

which is equivalent to

$$r \cdot a \leq r \cdot m(r) \leq r \cdot a.$$

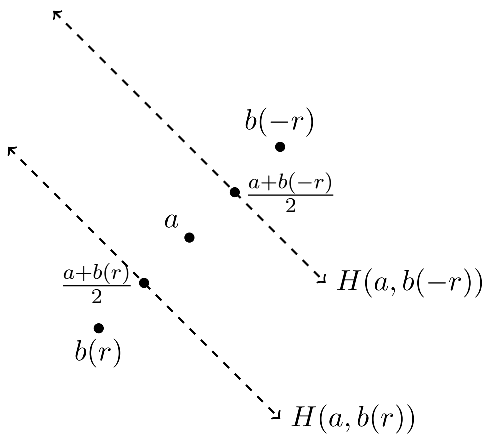

Figure 8: Policy a is majority preferred to $b(r)$ and $b(-r)$ if and only if $M(r)$ lies between $H(a, b(r))$ and $H(a, b(-r))$.

It is therefore necessary in the classic spatial model that the median hyperplane in direction r passes through a. Moreover, $GS(a, \beta) = \{a\}$ because $b(r) = b(-r) = a$ for all directions. Hence, a total median at a is necessary for a to be a Condorcet Winner.

For the same reason, the same conclusions hold if $a \in \text{int}(\beta)$ even if α and β are not all of $\mathbb{R}^n$. A similar conclusion holds if $a \in \text{bd}(\beta)$. In that case, in order for a to be a Condorcet Winner, it is necessary that there be an "almost" total median in which all median hyperplanes must either support a or be further from β in their respective directions. These kinds of situations are explored in detail in Matthews (1980), and so they are not considered further here.

In order for the constraints on the median hyperplanes implied by the model with disjoint policy spaces to be weaker than those in the classic spatial model, it must be the case that

$$r \cdot \frac{a + b(r)}{2} \leq r \cdot m(r) \leq r \cdot \frac{a + b(-r)}{2}$$

defines a non-degenerate interval. If, as we have assumed, the policy spaces α and β are disjoint, then neither $b(r)$ nor $b(-r)$ can equal a. Instead, there must be a positive distance from a to $b(r)$ and to $b(-r)$ and, hence, there must be a positive distance from a to $H(a, b(r))$ and $H(a, b(-r))$. Consequently,

$$r \cdot \frac{a + b(r)}{2} < r \cdot a < r \cdot \frac{a + b(-r)}{2},$$

as illustrated in Figure 8.

While a median hyperplane through a satisfies this condition, because in that case $r \cdot m(r) = r \cdot a$, this is not required. Instead, it is only required that the median hyperplane have a value of $r \cdot m(r)$ in the interval $[r \cdot \frac{a+b(r)}{2}, \frac{a+b(-r)}{2}]$, an interval containing a with a nonempty interior. Equivalently, $M(r)$ need only lie in $H(a, b(r))^+ \cap H(a, b(-r))^+$, a set with nonempty interior. Hence, even if all directions are vulnerable, policy a is an RCW if all median hyperplanes pass through non-degenerate regions containing a.

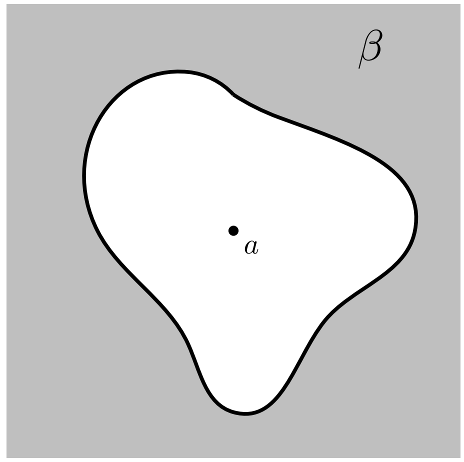

Figure 9: The gray set is an example of β such that all directions are vulnerable at policy a.

Theorem 3. If α and β are disjoint and all directions are vulnerable for policy a, then a is an RCW if and only if $\forall r \in \omega(a, \beta)$, $M(r) \in H(a, b(r))^+ \cap H(a, b(-r))^+$, which is a set containing a that has a nonempty interior.

Theorem 3 assumes that all directions are vulnerable, which implies that for any possible direction from policy a, there is a policy $b \in \beta$ which lies in that direction, as illustrated in Figure 9. This is not a natural assumption. However, it implies that the disjointness of α and β is sufficient to weaken the requirements for a to be an equilibrium, without making further restrictions on where alternatives to a may be proposed.

We now consider such a restriction. If β is convex in addition to our maintained assumptions, then at most half of the possible directions are vulnerable. In particular, if a given direction r is vulnerable, then $-r$ cannot be vulnerable as well; otherwise, a lies on a line connecting two policies in β and must therefore be in β.

Lemma 6. If β is convex, then $r \in \omega(a, \beta) \implies -r \notin \omega(a, \beta)$.

When r is a vulnerable direction but $-r$ is not, then $b(-r)$ does not exist. Consequently, by Lemma 3, the requirement for a to be an RCW is that $M(r) \subset H(a, b(r))^+$. There is not a constraint on $M(r)$ for the direction $-r$ to consider. Therefore, if it is assumed that β is convex, the conditions for the existence of an RCW are weakened compared to the case in which convexity is not assumed in two ways: (i) there are fewer directions that are vulnerable and (ii) for each direction that is vulnerable, the requirement is that the median hyperplane for that direction lies in a given halfspace; it need not pass through a specific point or subset of that halfspace.

Corollary 1. If β is convex, then at most half of all directions are vulnerable and, further, a is an RCW if and only if for each vulnerable direction, $M(r) \subset H(a, b(r))^+$.

The differences in the conditions for an equilibrium in each of the three cases discussed are summarized in Table 1.

Policy Spaces	Vulnerable directions	Median constraint for each vulnerable direction
$\beta = \mathbb{R}^n$	All	$M(r)$ must pass through a
$\alpha \cap \beta = \emptyset$ and β is non-degenerate and closed	All	$M(r)$ must lie in a set with nonempty interior which contains a.
$\alpha \cap \beta = \emptyset$ and β is non-degenerate, closed, and convex	At most half	$M(r)$ must lie in a halfspace containing a

Table 1: A summary of the conditions for a to be an RCW.

5.2 Existence of an Equilibrium

Aside from the question of whether a given policy a is an RCW, we also want to know under what conditions there exists any policy a in α such that a is an RCW. In particular, we want to know if an RCW exists in the model considered here when an equilibrium does not exist in the classic spatial model. This occurs in the classic spatial model if there is not a total median. In those cases, the yolk, introduced by McKelvey (1986), is used to summarize the locations of the median hyperplanes for a given distribution of voter ideal points. Specifically, a *yolk* is a hypersphere of minimal radius that is either tangent to or intersects all median hyperplanes.

Definition 8. A *yolk* is a hypersphere with center $c \in \mathbb{R}^n$ and radius $\tau \in \mathbb{R}_+$, denoted $Y(c, \tau)$, such that $Y(c, \tau) \cap M(r) \neq \emptyset \; \forall r$, and such that for any $\tau' < \tau$, there exists a direction r such that $Y(c, \tau) \cap M(r) = \emptyset$.

In this definition, the requirement that $Y(c, \tau) \cap M(r) \neq \emptyset$ must be satisfied for every possible direction, not just directions that are vulnerable. As noted by Martin et al. (2016), while the center c may not be unique, the radius τ is. If a yolk $Y(c, \tau)$ exists that is contained in $GS(a, \beta)$, then a is an RCW because all median hyperplanes either support $GS(a, \beta)$ or are strictly further from β. If $Y(c, \tau)$ is contained in the interior of $GS(a, \beta)$, then a is an SRCW.

Lemma 7. If $Y(c, \tau) \subset GS(a, \beta)$, then a is an RCW. If $Y(c, \tau) \subset \text{int}GS(a, \beta)$, then a is an SRCW.

In the classic model of spatial voting in $\mathbb{R}^n$, an equilibrium exists if and only if $\tau = 0$; that is, if and only if there is a total median. In the model considered here, if any $Y(c, \tau)$ lies within $GS(a, \beta)$ for at least one $a \in \alpha$ or within $GS(b, \alpha)$ for at least one $b \in \beta$, then an equilibrium exists even if $\tau > 0$.

We now consider when this is the case. Let $a^*(y)$ be the closest policy to y in α and let $b^*(y)$ be the closest policy to y in β. Proposition 1 implies that a yolk is contained in $GS(a, \beta)$ for at least one a in α if there exists an $a \in \alpha$ such that $||y - a|| \leq ||y - b^*(y)||$ for every $y \in Y(c, \tau)$. This must certainly hold if

$$\max_{y \in Y(c,\tau)} ||y - a|| \leq \min_{y \in Y(c,\tau)} ||y - b^*(y)||.$$

Note that the solution value on the left hand side of this inequality need not equal the value on the right hand side. The maximum distance of a point in the yolk from a is $||c - a|| + \tau$. The minimum distance of a point in a yolk to the closest policy to it in β is $||c - b^*(c)|| - \tau$. Consequently, the yolk is contained in $GS(a, \beta)$ if

$$||c - a|| + \tau \leq ||c - b^*(c)|| - \tau$$

or equivalently, if

$$||c - a|| \leq ||c - b^*(c)|| - 2\tau.$$

Candidate A can minimize the left hand side of this inequality by choosing $a = a^*(c)$. Intuitively, this inequality says that if one candidate can get at least 2τ closer to the center of a yolk than her opponent, then there is an RCW in that candidate's policy set. The preceding argument establishes the following theorem:

Theorem 4. Let $Y(c, \tau)$ be a yolk for the distribution of voter ideal points f. Then, an RCW exists if

$$||c - a^*(c)|| \leq ||c - b^*(c)|| - 2\tau$$

or if

$$||c - a^*(c)|| \geq ||c - b^*(c)|| + 2\tau.$$

We may gain some intuition by considering what happens when $\tau = 0$. In this case, we can think of c as the ideal point of a decisive voter, and whichever candidate that voter strictly prefers is the winner of the election. Similarly, in the one dimensional case, the winner is the candidate who the median voter prefers. Now suppose that $\tau > 0$. If one candidate can propose a policy 2τ closer to c than her opponent, then she wins for sure. In a rough way, the intuition that office motivated candidates want to appeal to a centrally located voter, prominent in one dimensional competition, is maintained.

Example (Continued). Returning to our running example, using Theorem 4, an RCW exists with A as the winner in this example if there exists a policy $a \in \alpha$ that is closer to the center c of the yolk $Y(c, \tau)$ by at least 2τ than the closest $b \in \beta$ is to c. Let c_x be the x coordinate of c and let c_y be its y coordinate. Let $b^*(c)$ be the closest policy $b \in \beta$ to c. Consider first the case in which $c_x \leq \varepsilon$ and $c_y \geq \varepsilon$, which is Region I in Figure 10. Then $b^*(c)$ is either $(c_x, -\varepsilon)$ or $(-\varepsilon, -\varepsilon)$. The distance from c to $b^*(c)$ for c in Region I is bounded below by $c_y + \varepsilon$, which occurs when c is directly above the upper boundary of β. If the closest policy is $(-\varepsilon, -\varepsilon)$, then the distance from c to $b^*(c)$ is the hypotenuse of a triangle with one leg equal to $c_y + \varepsilon$, and so this distance is strictly greater than $c_y + \varepsilon$. For candidate A, $a^*(c) = (c_y, \varepsilon)$. The distance from c to $a^*(c)$ is therefore $-c_x + \varepsilon$. Hence, c is closer to $a^*(c)$ than $b^*(c)$ by 2τ if $c_y + \varepsilon > c_x + \varepsilon + 2\tau$ or, equivalently, if $c_y > -c_x + 2\tau$.

If $c_x \geq \varepsilon$ and $c_y \leq \varepsilon$, then c is in Region II. In that case, $b^*(c)$ is either $(c_y, -\varepsilon)$ or $(-\varepsilon, -\varepsilon)$. By an analogous argument, the distance from c to $b^*(c)$ is no less than $c_x + \varepsilon$. For candidate A, $a^*(c) = (c_x, \varepsilon)$ and, hence, the distance from c to $a^*(c)$ is $-c_y + \varepsilon$. Hence, c is closer to $a^*(c)$ than $b^*(c)$ by 2τ if $-c_y + \varepsilon < c_x + \varepsilon - 2\tau$ or, equivalently, if $c_y > c_x + 2\tau$.

If $c_x \geq \varepsilon$ and $c_y \geq \varepsilon$, then $c \in \alpha$ and $a^*(c) = c$, and therefore the distance from c to $a(c)$ is 0. For candidate B, $b^*(c) = (-\varepsilon, -\varepsilon)$, in which case c is closer to $a^*(c)$ than to $b^*(c)$ by at least 2τ if c is at least 2τ from $(-\varepsilon, -\varepsilon)$.

These three inequalities establish the rightmost dashed curve in Figure 11. If c is above this curve, there is an RCW in α. By a symmetric argument, if c is below the leftmost dashed curve, there is an RCW in β. Hence, there is an RCW for one of A or B if the center of a yolk lies in one of two halfspaces defined by the two lines described above and is at least 2τ from the corner of the opponent's policy set. Even when c is in the area between the dashed curves, an equilibrium may exist, as Theorem 4 only provides a sufficient condition for the yolk to be contained in $GS(a, \beta)$, which itself is only a sufficient condition for the existence of an equilibrium.

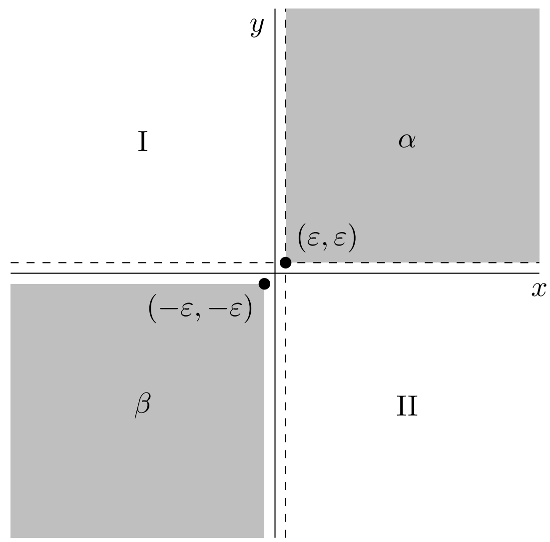

Figure 10: Possible locations for the center of a yolk.

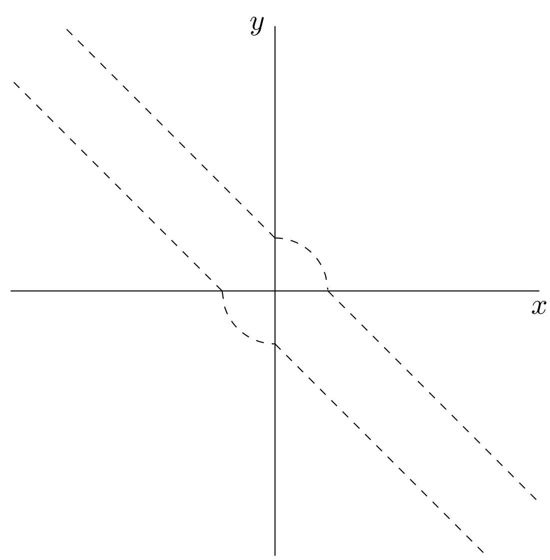

Figure 11: If there is a yolk with center c located above the upper dashed curve, then there exists $a \in \alpha$ such that a is an RCW, and similarly for B if c is below the lower dashed curve.

6 Robustness to Perturbations of the Distribution

An equilibrium in the classic spatial model is extremely fragile. Changing voters' ideal points even infinitesimally can result in the nonexistence of equilibrium (Rubinstein, 1979). In this section, we show that equilibria in our model are robust to two types of changes in the distribution of the voters' ideal points. We show that if a is an RCW, then there are some directions in which voters can be moved by an arbitrary distance while preserving a as an RCW. We also show that if a is an SRCW, then we can move voters ideal points in an arbitrary direction by a small distance while preserving a as an SRCW. In this model, not only does an RCW continue to exist if voter ideal points are moved, but it is the same policy as with the original distribution. To prove these results, we first establish in what direction and by what magnitude median hyperplanes move when some of the density of voters is moved in the direction r^*.

A distribution of voters $\bar{f}$ formed by moving density in the direction r^* relative to f has median hyperplanes that are further from β for any direction r such that $r \cdot r^* \geq 0$. For intuition, first consider the case in which there are a finite number of voter ideal points. If a voter's ideal point is moved in direction r^* by the magnitude $\delta > 0$, the projection of the voter's ideal point onto direction r is non-decreasing in δ for any r such that $r \cdot r^* \geq 0$. By definition,

$$\theta \in M(r)^+ \iff \theta \cdot r \geq m(r) \cdot r.$$

Thus, for an ideal point shifted by δ in direction r^*,

$$(\theta + \delta r^*) \cdot r = \theta \cdot r + \delta r^* \cdot r \geq m(r) \cdot r$$

because $r^* \cdot r \geq 0$. Hence, if such a voter has an ideal point in $M(r)^+$, the voter's shifted ideal point is also in $M(r)^+$. Therefore, the median hyperplane for direction r for the distribution $\bar{f}$ is either coincident with $M(r)$ or further from β. This situation is illustrated in Figure 12.

For a direction r such that $r \cdot r^* < 0$, it may be the case that the median hyperplane for direction r is closer to β for $\bar{f}$ than for f. This change occurs only if the voter's ideal point is moved out of $M(r)^+$, so that fewer than half of the voters are contained in $M(r)^+$. In that case, if a hyperplane orthogonal to r^* contains this voter's shifted ideal point, then we know that that at least half of the voters are in the positive halfspace of this hyperplane. Consequently, the new median hyperplane for direction r, $\bar{M}(r)$, must either be coincident with this hyperplane or be contained in its positive halfspace. Thus, the furthest the median hyperplane in direction r must be adjusted is δ. This situation is illustrated in Figure 13.

In the continuous case, the analogous modification to the voter ideal points involves moving a mass of ideal points. Specifically, two sets of positive measure in $\mathbb{R}^n$, Γ and $\bar{\Gamma}$, are chosen such that $\bar{\Gamma}$ is a linear transformation of Γ in direction r^* by some distance δ. Then, we reduce the density on the set Γ by a positive ϵ and increase the density on $\bar{\Gamma}$ by ϵ.

Definition 9. Let Γ be a set of positive measure in $\mathbb{R}^n$. Let $\bar{\Gamma} = \Gamma + \delta r^*$, where $\delta > 0$ and $r^* \in \mathbb{R}^n$ is a unit vector. Choose $\epsilon > 0$ such that $\epsilon < f(x),\ \forall x \in \Gamma$. Let

$$\bar{f}(x) = \begin{cases} f(x) - \epsilon & \text{if } x \in \Gamma \\ f(x) + \epsilon & \text{if } x \in \bar{\Gamma} \\ f(x) & \text{if } x \notin \Gamma \cup \bar{\Gamma}. \end{cases}$$

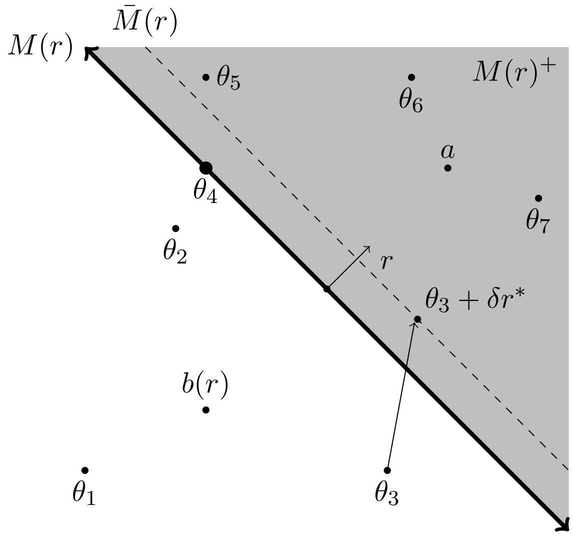

Figure 12: If a voter is moved the direction r^*, then the median hyperplane for r cannot move toward $b(r)$.

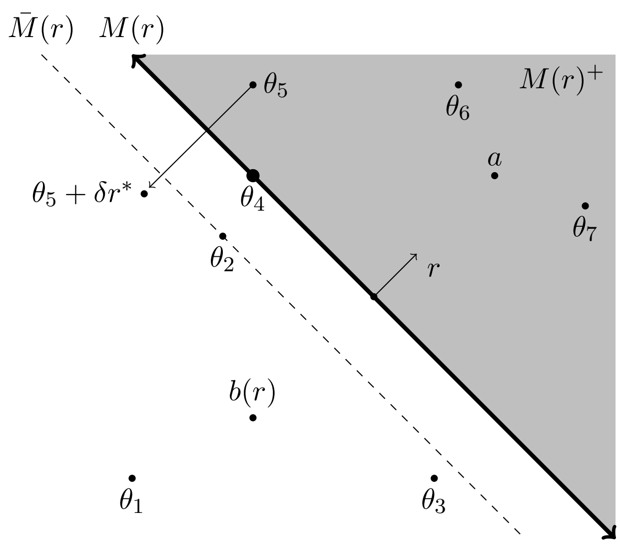

Figure 13: If a voter is moved in the direction r^*, then $\bar{M}(r)$ is no further than δ from $M(r)$. In this example, $r^* = -r$.

We first prove that $\bar{f}$ is strictly positive on $\mathbb{R}^n$ and that the integral of $\bar{f}$ over $\mathbb{R}^n$ is one, so that $\bar{f}$ is a probability density function.

Lemma 8. The function $\bar{f}$ is everywhere strictly positive, and $\int_{\mathbb{R}^n} \bar{f}(x) = 1$. Therefore, $\bar{f}$ is a probability density function.

Let $\{M(r) - dr\} \equiv \{x \in \mathbb{R}^n | r \cdot x = r \cdot m(r) - d\}$ be a shift of the median hyperplane $M(r)$ in the direction $-r$ by the distance d. When $d = 0$, this hyperplane is equal to $M(r)$. Consider the integral over its positive halfspace, $\int_{\{M(r)-dr\}^+} \bar{f}(x)$. As d increases, this hyperplane moves further in direction $-r$, thereby expanding the set $\{M(r) - dr\}^+$ to include more of the strictly positive density of voters. Thus, the integral is strictly increasing in d.

Lemma 9. The value of $\int_{\{M(r)-dr\}^+} \bar{f}(x)$ is strictly increasing in d.

Let $\bar{M}(r)$ be the median hyperplane for direction r when voter ideal points are distributed according to $\bar{f}$. Because $M(r)$ and $\bar{M}(r)$ have the same normal vector, there must exist a scalar d such that $\bar{M}(r) = M(r) - dr$. Hence, $\bar{M}(r) \subset M(r)^+$ if and only if d is nonpositive. A positive value of d indicates that $\bar{M}(r)$ is not weakly further from β than $M(r)$. We show in Lemma 10 that this can only happen if $\int_{M(r)^+} \bar{f}(x) < \frac{1}{2}$.

Lemma 10.

$$\bar{M}(r) \not\subset M(r)^+ \iff \int_{M(r)^+} \bar{f}(x) < \frac{1}{2}.$$

For any r such that $r \cdot r^* \geq 0$, there is either no change in the density of voters in $M(r)^+$ or an increase, so by Lemma 10 the median hyperplane $\bar{M}(r)$ is weakly further from β than $M(r)$. For any r such that $r \cdot r^* < 0$, it is possible that there is less density of voters in $M(r)^+$ when ideal points are distributed according to $\bar{f}$ rather than f, so it may be the case that $\bar{M}(r) \not\subset M(r)^+$. However, we can bound the distance of $\bar{M}(r)$ from $M(r)$ by δ.

Proposition 3. For any direction r for which $r \cdot r^* \geq 0$, $\bar{M}(r) \subset M(r)^+$. For any direction r for which $r \cdot r^* < 0$, the distance between $\bar{M}(r)$ and $M(r)$ is at most δ.

Consider a policy a and let $\omega(a, \beta)^+ = \{k \in \mathbb{R}^n | k \cdot r \geq 0, \forall r \in \omega(a, \beta)\}$. By Proposition 3, for any $r^* \in \omega(a, \beta)^+$, if any subset of voters is moved in the direction r^* by any positive amount, then the median hyperplanes for every vulnerable direction would either remain the same or move further from β. Hence, if a is an RCW for the distribution f, it is an RCW for the distribution $\bar{f}$ as well. If policy a is an SRCW for the distribution f, then $M(r) \subset H(a, b(r))^{++}$ for every vulnerable direction. Consider a distribution $\bar{f}$ formed by a movement of any subset of the voters in any direction r^*. Because the median hyperplanes move by at most δ for any direction for which $r \cdot r^* < 0$, by making δ sufficiently small, it is ensured that $\bar{M}(r) \subset H(a, b(r))^{++}$. Hence, a is an SRCW for $\bar{f}$ as well.

Theorem 5. For any policy $a \in \alpha$ and any change of the density of voter ideal points from f to $\bar{f}$, as described in Definition 9:

1. If a is an RCW when the voter ideal points are distributed according to f, then for any $r^* \in \omega(a, \beta)^+$ and $\delta > 0$, a is an RCW when voters have ideal points distributed according to $\bar{f}$.

2. If a is an SRCW when the voter ideal points are distributed according to f, then for any r^* and δ sufficiently small, a is an SRCW when voters have ideal points distributed according to $\bar{f}$.

Theorem 5 shows that an equilibrium in the model with disjoint policy sets is more robust than in the classic spatial model. Because the existence of an equilibrium in the classic model depends upon the existence of a total median, moving even a small amount of density of the voter ideal points an infinitesimal distance can cause the equilibrium to no longer exist. In our model, because candidate B is limited in what policies she can propose, moving voters ideal points away from the policies B can offer and toward a does not reduce support for a. Further, even if voter opinion moves marginally away from a and toward β in some directions, if a is winning with a strict majority over any policy B could offer in the initial distribution, while that majority may be less in the new distribution, a could remain a winning policy.

However, Theorem 5 also implies that an RCW need not coincide with any measure of centrality, be it the mean, the center of the yolk, or a total median if one exists. Suppose policy a is an RCW. Even if a is coincident with some measure of centrality, by Theorem 5, we can perturb the distribution in such a way that moves the center of the yolk, the mean, or the total median if one exists, and a remains an RCW. This conclusion is in contrast to the classic spatial model, in which an equilibrium, if one exists, must be located at the total median. Further, candidates in our model need not be responsive to changes in voter opinion. Policy a may remain an RCW even if voter ideal points move an arbitrary distance. Hence, stability may come at a cost of reduced responsiveness.

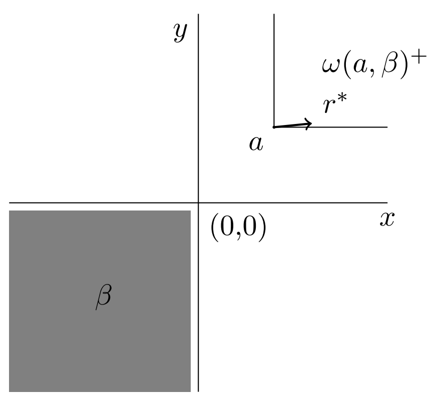

Figure 14: The set of directions $\omega(a,\beta)^+$ for which voters can be moved while preserving a as an RCW.

Example (Continued). We return to our running example to consider the implications of Theorem 5. Consider the set of vulnerable directions for any a in α. If b has an arbitrarily small x coordinate and a y coordinate of $-\varepsilon$, then r_{ab} is arbitrarily close to $(1,0)$. If b instead has an arbitrarily small y coordinate and an x coordinate of $-\varepsilon$, then r_{ab} is arbitrarily close to $(0,1)$. By convexity of β, any direction that is a positive combination of $(-1,0)$ and $(1,0)$ is also vulnerable. Consequently, $\omega(a,\beta)$ is the set of all directions with positive x and y components. Therefore, for any RCW $a \in \alpha$, the set $\{k \in \mathbb{R}^2_+ || ||k|| = 1\}$ is a subset of $\omega(a,\beta)^+$. Consider $r^* \in \mathbb{R}^2_+$. A shift of voter ideal points in direction r^* implies that voters are moving toward more Republican policies on both gun rights and defense spending, at least weakly. As illustrated in Figure 14, such a direction r^* lies in $\omega(a,\beta)^+$. Therefore, by Theorem 5, a remains an RCW if any subset of the voters ideal points are moved by any distance in direction r^*. Hence, a shift in voters' opinions toward more Republican policies does not decrease the support for any given Republican policy. This observation is in contrast to the classic spatial model, in which both candidates are forced to move to the new total median, should one exist, in order to avoid losing with certainty.

7 Conclusion

The classic spatial model of political competition assumes that candidates may propose any policy in a Euclidean space and, as a result, an equilibrium does not exist unless the distribution of voter ideal points satisfies very strong conditions. Specifically, the gradients of the voters' utility functions must satisfy Plott's pairwise symmetry conditions. In this paper, we have shown that when candidates have disjoint policy spaces, an equilibrium exists with much less restrictive assumptions on the distribution of voter ideal points. In the classic spatial model, because the existence of an equilibrium is robust only to perturbations of the voters' ideal points that preserve pairwise symmetry, it is difficult to develop comparative statics with respect to the distribution of the voter ideal points. In our model with disjoint policy sets, because candidates can win with more than half of the votes, small perturbations of voter ideal points need not result in the nonexistence of an equilibrium. Further, because the directions in which competing policies may be located is limited, large perturbations in which voters are moved by an arbitrary distance need not result in the nonexistence of an equilibrium either. Importantly, not only does an equilibrium exist even when voters' preferences have been perturbed, it is the same policy as before.

In this model, two forces account for the existence of an equilibrium with weaker distributional assumptions. First, a candidate is unable to get arbitrarily close to her opponent's policy and, therefore, infinitesimal deviations from an equilibrium are impossible. Hence, it need not be the case that an equilibrium policy be a total median. Instead, median hyperplanes only need to be close enough to an equilibrium policy. Second, a candidate with a policy that lacks majority support may not be able to outflank her opponent because such a maneuver is not feasible. In particular, because not every proposed deviation from a candidate's policy is feasible, some median hyperplanes are not relevant to determining if her opponent's policy can be defeated, and can therefore be ignored.

We have shown that political parties help to stabilize political competition by keeping candidates' proposed policies from converging. By contrast, a candidate with no ideological commitments is potentially destabilizing because she may propose alternative policies arbitrarily close to any policy proposed by her opponent in any direction.

Our results are dependent upon the assumption that all voters have Euclidean preferences. Because policies cannot be arbitrarily close in our analysis, we must directly consider the upper contour sets of the voters' utility functions to find policies that are preferred, rather than their gradients. Unless these sets exhibit considerable structure, as is the case with Euclidean preferences, it is unlikely that tractable conditions for the existence of an equilibrium can be derived for large numbers of voters. Consequently, it may be worthwhile to find an alternative way of modeling the structure of the candidates' permissible policies mandated by the parties. A natural extension of our analysis would consider the case in which, rather than a hard constraint, the degree of party support for a candidate diminishes continuously as she moves away from the approved party policies. For example, individuals who actively participate in an electoral campaign may not work as enthusiastically to promote a candidate if she becomes more ideologically distant from their preferred policies, thereby diminishing this candidate's valence. With such an extension, a gradient based characterization of an equilibrium may be possible for a more general class of preferences, especially if combined with a probabilistic voting model.

Appendix

Proof of Lemma 1. Let γ^* solve $b(r) = a - \gamma r$. Because $||a - \theta|| \leq ||b(r) - \theta||$, $a \in H(a, b(r))^+$. Therefore, $r \cdot \theta \geq r \cdot \frac{a+a-\gamma^* r}{2}$ or, equivalently, $r \cdot \theta \geq r \cdot a - \frac{\gamma^*}{2}$ because r has unit length. For any $b \in \beta(r)$ such that $b \neq b(r)$, the associated γ is necessarily larger than γ^*. Because $r \cdot \theta \geq r \cdot a - \frac{\gamma^*}{2} \implies r \cdot \theta > r \cdot a - \frac{\gamma}{2}$, it must be that $\theta \in H(a, b)^{++}$. Hence, θ is strictly closer to a than b for every $b \in \beta(r)$ for which $b \neq b(r)$. □

Proof of Proposition 1. (i) If $\theta \in GS(a, \beta)$ then $\theta \in H(a, b(r))^+$, $\forall r \in \omega(a, \beta)$. Therefore, by Lemma 1, for every $r \in \omega(a, \beta)$, $||a - \theta|| \leq ||b - \theta||$, $\forall b \in \beta$. If $\theta \notin GS(a, \beta)$, then there exists an r such that $\theta \notin H(a, b(r))^+$, which implies for this r that $||a - \theta|| > ||b(r) - \theta||$.

We now prove the sufficiency parts of (ii) and (iii).

(ii) If $\theta \in \text{int}GS(a, \beta)$, then $\theta \in H(a, b(r))^{++}$ and therefore $||a - \theta|| < ||b(r) - \theta||$, $\forall r \in \omega(a, \beta)$. Thus, by Lemma 1, $||a - \theta|| < ||b - \theta||$, $\forall b \in \beta$.[7]

(iii) If $\theta \in \text{bd}(GS(a, \beta))$, then by (i), $||a - \theta||^2 \leq ||b - \theta||^2$, $\forall b \in \beta$. We now show that it must be the case that $\theta \in H(a, b(r))$ for at least one r. Suppose, contrary to our claim, that $\theta \in H(a, b(r))^{++}$, $\forall r \in \omega(a, \beta)$. Then $r \cdot \theta > r \cdot \frac{a+b(r)}{2}$, $\forall r \in \omega(a, \beta)$. Choose any $\epsilon > 0$, and consider any point x of distance ϵ from θ. Let the direction pointing from θ to x be r'. Note that for x,

$$r \cdot x = r \cdot (\theta + \epsilon r') = r \cdot \theta + \epsilon r \cdot r' \geq r \cdot \theta - \epsilon, \ \forall r \in \omega(a, \beta),$$

because $r \cdot r' \geq -1$. Thus, for $\epsilon < \bar{\epsilon} \equiv \min_{r \in \omega(a,\beta)} r \cdot \theta - r \cdot \frac{a+b(r)}{2}$,

$$r \cdot \theta - \epsilon > r \cdot \frac{a + b(r)}{2}, \ \forall r \in \omega(a, \beta).$$

Hence, $x \in H(a, b(r))^{++}$, $\forall r \in \omega(a, \beta)$. Because this holds for an arbitrary x within $\bar{\epsilon}$ of θ, we have shown that an open ball around θ is contained in $GS(a, \beta)$, contradicting our assumption that θ is a boundary point. Hence, $\theta \in H(a, b(r))$ for at least one $r \in \omega(a, \beta)$, and thus, $||a - \theta|| = ||b(r) - \theta||$ for that r.

We now prove the necessary parts of (ii) and (iii).

(ii) Suppose $\theta \notin \text{int}GS(a, \beta)$. Then either $\theta \notin GS(a, \beta)$ or $\theta \in \text{bd}(GS(a, \beta))$. If $\theta \notin GS(a, \beta)$, then there must be at least one vulnerable direction r for which $\theta \notin H(a, b(r))^+$, and therefore $||b(r) - \theta|| < ||a - \theta||$. If $\theta \in \text{bd}(GS(a, \beta))$, then the proof of sufficiency for (iii) proves that $||a - \theta|| = ||b(r) - \theta||$ for some $r \in \omega(a, \beta)$. Thus, if $\theta \notin \text{int}GS(a, \beta)$, $||a - \theta||^2 \geq ||b - \theta||^2$ for at least one $b \in \beta$.

[7] The interior of of an infinite intersection is only a subset of the intersection of the interiors. They are not necessarily equal. This argument relies only on the fact that $\text{int}GS(a, \beta) \subset \cap_{r \in \omega} H(a, b(r))^{++}$.

(iii) Consider an ideal point θ such that $||a - \theta|| = ||b - \theta||$ for some $b(r)$ and $||a - \theta||^2 \leq ||b - \theta||^2$, $\forall b \in \beta$. Clearly, $\theta \in GS(a, \beta)$ by the proof of (i). Notice that $\theta \notin \text{int}GS(a, \beta)$, because that is only possible if $||a - \theta||^2 < ||b - \theta||^2$, $\forall b \in \beta$ by (ii). Because $GS(a, \beta)$ is a closed set, θ must therefore be on the boundary of $GS(a, \beta)$. □

Proof of Lemma 2. Policy a is majority preferred to b if and only if

$$\int_{H(a,b)^+} f(x) \geq 1/2.$$

If $M(r_{ab}) \subset H(a, b)^+$, because at least half of the measure of voters is in $M(r_{ab})^+$, at least half of the measure of voters is in $H(a, b)^+$, so $a \succeq b$. Because the distribution of ideal points has a positive density on all of $\mathbb{R}^n$, if $M(r_{ab}) \subset H(a, b)^{++}$, then there must be a positive measure of voters between $M(r_{ab})$ and $H(a, b)$ and, hence, strictly more than half of the measure of voters must be in $H(a, b)^+$. Finally, if $M(r_{ab}) \subset H(a, b)^-$, then it must be that at least half of the measure of voters is closer to b than a, so $b \succeq a$, whereas if $M(r_{ab}) \not\subset H(a, b)^+$, then a symmetric argument to the case of $H(a, b)^{++}$ implies $b \succ a$. Because this exhausts all possible locations of $M(r_{ab})$, the result is shown. □

Proof of Lemma 3. For weak preference, we have $a \succeq b(r) \iff M(r) \subset H(a, b(r))^+$ by Lemma 2. Also, $M(r) \subset H(a, b(r))^+ \iff r \cdot \frac{a+b(r)}{2} \leq r \cdot m(r)$. For any $b \in \beta(r)$ such that $b \neq b(r)$, $r \cdot \frac{a+b}{2} < r \cdot \frac{a+b(r)}{2}$, and therefore $r \cdot \frac{a+b}{2} < r \cdot m(r)$. Thus, $M(r) \subset H(a, b)^{++}$, so that $a \succ b$. Hence, $M(r) \subset H(a, b(r))^+ \implies a \succeq b, \forall b \in \beta(r)$. Conversely, if $M(r) \not\subset H(a, b(r))^+$, then $a \not\succeq b(r)$ by Lemma 2. Therefore, $a \succeq b, \forall b \in \beta(r) \iff M(r) \subset H(a, b(r))^+$.

For strict preference, $a \succ b(r) \iff M(r) \subset H(a, b(r))^{++}$ by Lemma 2. For any $b \in \beta(r)$ such that $b \neq b(r)$, $r \cdot \frac{a+b}{2} < r \cdot \frac{a+b(r)}{2}$, and therefore $r \cdot \frac{a+b}{2} < r \cdot m(r)$. Thus, $M(r) \subset H(a, b)^{++}$, so that $a \succ b$. Hence, $M(r) \subset H(a, b(r))^{++} \implies a \succ b, \forall b \in \beta(r)$. Conversely, if $M(r) \not\subset H(a, b(r))^{++}$, then $a \not\succ b(r)$ by Lemma 2. Therefore, $a \succ b, \forall b \in \beta(r) \iff M(r) \subset H(a, b(r))^+$. □

Proof of Proposition 2. By Lemma 3, $a \succeq b$ for every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^+$. Therefore, $a \succeq b$ for every $b \in \beta(r)$ if and only if $M(r)$ is weakly further from β than $H(a, b(r))$. Hence, $a \succeq b$ for every $b \in \beta$ if and only if $M(r)$ is weakly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$.

Further, by Lemma 3, $a \succ b$ for every $b \in \beta(r)$ if and only if $M(r) \subset H(a, b(r))^{++}$. Therefore, $a \succ b$ for every $b \in \beta(r)$ if and only if $M(r)$ is strictly further from β than $H(a, b(r))$. Hence, $a \succ b$ for every $b \in \beta$ if and only if $M(r)$ is strictly further from β than $H(a, b(r))$ for every $r \in \omega(a, \beta)$. □

Proof of Theorem 1. For every $r \in \omega(a, \beta)$, $GS(a, \beta) \subseteq H(a, b(r))^+$, so $S(r) \subset H(a, b(r))^+$. Hence, $M(r) \subset S(r)^+ \implies M(r) \subset H(a, b(r))^+$. Because this holds for every $r \in \omega(a, \beta)$, by Proposition 2, a is an RCW.

If $M(r) \subset S(r)^{++}$ for every $r \in \omega(a, \beta)$, then $r \cdot x > r \cdot \frac{a+b(r)}{2}$, $\forall x \in M(r)$, which implies $M(r) \subset H(a, b(r))^{++}$. Because this holds for every $r \in \omega(a, \beta)$, by Proposition 2, a is an SRCW. □

Proof of Lemma 4. By the definition of a supporting hyperplane, x^* must be a solution to:

$$\min_x r \cdot x \text{ subject to } x \in GS(a, \beta).$$

Suppose, contrary to our claim, that $r \cdot x^* > r \cdot \frac{a+b(r)}{2}$. By assumption, $S(r)$ is the unique supporting hyperplane to $GS(a,\beta)$ at x^*, so no hyperplane of equidistance can be supporting. Therefore, $r' \cdot x^* > r' \cdot \frac{a+b(r')}{2}, \forall r' \in \omega(a,\beta)$.

Now consider the point $x^* - \delta r$, where $\delta > 0$. Observe that

$$r' \cdot x^* - \delta r \cdot r' > r' \cdot x^* - \delta,\ \forall r' \in \omega(a,\beta)$$

because $r \cdot r' \leq 1$. Hence, for a sufficiently small $\delta > 0$,

$$r' \cdot x^* - \delta > r' \cdot \frac{a+b(r')}{2},\ \forall r' \in \omega(a,\beta).$$

Thus, $x^* - \delta r$ is in $GS(a,\beta)$. Note that for direction r,

$$r \cdot (x^* - \delta r) = r \cdot x^* - \delta$$

because r has unit length. Clearly, $r \cdot x^* - \delta < r \cdot x^*$, which contradicts the assumption that x^* is on a supporting hyperplane of $GS(a,\beta)$.

Hence, $S(r) = H(a,b(r))$. Because $M(r)$ is a hyperplane with the same normal vector as $H(a,b(r))$, if $M(r)$ supports $GS(a,\beta)$, it must be coincident with $H(a,b(r))$. If it is further from β than $H(a,b(r))$, then $M(r) \subset H(a,b(r))^{++}$. Because $a \succeq b$ for every $b \in \beta(r)$ if and only if $M(r) \subset H(a,b(r))^{+}$ by Lemma 3, it is therefore necessary for $M(r)$ to be weakly further from β than $S(r)$. □

Proof of Lemma 5. Consider a point $x \in \text{bd}(GS(a,\beta))$ which is supported by more than one hyperplane. Because a convex set is the intersection of its supporting hyperplanes, we know that the supporting hyperplanes at x are hyperplanes of equidistance. Consider two of the supporting hyperplanes, chosen arbitrarily if there are more than two. Because x is on both hyperplanes, it must be equidistant to two points in β, say b_1 and b_2, and that distance must equal $||a - x||$ by Proposition 1. By the convexity of β, there is a unique closest point to x in β. If $||a-x||$ is the minimal distance from x to a point in β, this contradicts uniqueness. If it is not the minimal distance, then there exists another point b_3 that is strictly closer to x, and therefore $||b_3 - x|| < ||a - x||$. But then $x \notin GS(a,\beta)$, by Proposition 1. Therefore, $x \in GS(a,\beta)$ cannot have multiple supporting hyperplanes when β is convex. □

Proof of Theorem 2. Consider the solution x^* to

$$\min_x r \cdot x \text{ subject to } x \in GS(a,\beta)$$

for an arbitrarily chosen $r \in \omega(a,\beta)$. By Lemma 5, the solution x^* must lie on the unique hyperplane $S(r)$ to $GS(a,\beta)$ in direction r because $r \cdot x \geq r \cdot x^*$ for all $x \in GS(a,\beta)$. By Lemma 4, it is therefore necessary that $M(r)$ be weakly further from β than $S(r)$. Because this is true for every $r \in \omega(a,\beta)$, it is therefore necessary that $M(r)$ be weakly further from β than $S(r)$ for every $r \in \omega(a,\beta)$.

By Theorem 1, it is sufficient that $M(r)$ be weakly further from β than $S(r)$. □

Proof of Theorem 3. Necessity of $M(r) \subset H(a,b(r))^{+} \cap H(a,b(-r))^{+}$ for a to be an RCW is shown in the text preceding Theorem 3; sufficiency is implied by Proposition 2.

Now suppose that r and $-r$ are both in $\omega(a,\beta)$. Because $a \neq b(r)$, we know that $b(r) = a - \gamma_1 r$ for some $\gamma_1 > 0$. Similarly, because $a \neq b(-r)$, we know that $b(-r) = a + \gamma_2 r$ for some $\gamma_2 > 0$. We now show that $H(a,b(r))^+ \cap H(a,b(-r))^+$ has a nonempty interior. Specifically, we show that for $\epsilon \leq \min\{\gamma_1/2, \gamma_2/2\}$, an open ball of radius $\epsilon > 0$ centered at a, is contained in $H(a,b(r))^+ \cap H(a,b(-r))^+$.

For any x such that $||a-x|| < \epsilon$, we have that $r \cdot (a-x) < ||a-x||$ because r is a vector of unit length. Hence, $r \cdot (a-x) \leq \frac{\gamma_1}{2}$ or, equivalently, $r \cdot \frac{a+a-\gamma_1 r}{2} \leq r \cdot x$. Thus, using the definition of $b(r)$, $r \cdot \frac{a+b(r)}{2} \leq r \cdot x$. Hence, $x \in H(a,b(r))^+$. A symmetric argument shows that $r \cdot x \leq r \cdot \frac{a+b(-r)}{2}$, and, hence, that $x \in H(a,b(-r))^+$. □

Proof of Lemma 6. Assume, contrary to our claim, that there exists $r \in \omega(a,\beta)$ such that $-r \in \omega(a,\beta)$. Therefore, there exist distinct $b(r), b(-r) \in \beta$. But then a lies on a line connecting $b(r)$ and $b(-r)$. Because β is convex, $a \in \beta$, contradicting the assumption that $a \notin \beta$. □

Proof of Theorem 1. This theorem follows immediately from from Lemma 3 and Proposition 2. □

Proof of Lemma 7. Suppose $Y(c,\tau) \subset GS(a,\beta)$, but there exists an $r \in \omega(a,\beta)$ such that $M(r) \not\subset H(a,b(r))^+$. Then by the definition of $Y(c,\tau)$, there must exist a $y \in Y(c,\tau)$ such that $y \in M(r)$. But then $y \notin H(a,b(r))^+$, contradicting the assumption that $Y(c,\tau) \subset GS(a,\beta)$. Thus, $M(r) \subset H(a,b(r))^+, \forall r \in \omega(a,\beta)$, and by Lemma 3, a is an RCW. By a parallel argument, if $Y(c,\tau) \subset \text{int}GS(a,\beta)$ then $M(r) \subset H(a,b(r))^{++}, \forall r \in \omega(a,\beta)$, and by Lemma 3, a is an SRCW. □

Proof of Theorem 5. The proof is in the text preceding Theorem 5. □

Proof of Lemma 8. First, we show that $\bar{f}$ is positive. For every $x \in \Gamma$, $\bar{f}(x) > 0$ because $f(x) > \epsilon$. For every $x \in \bar{\Gamma}$, $\bar{f}(x) > 0$ because $\epsilon > 0$ and $f(x) > 0$. For every $x \notin \Gamma \cup \bar{\Gamma}$, $\bar{f}(x) > 0$ because $f(x) > 0$. Therefore, $\bar{f}(x) > 0$, $\forall x \in \mathbb{R}^n$.

Second, we show that $\int_{\mathbb{R}^n} \bar{f}(x) = 1$. Because $\bar{\Gamma} = \Gamma + \delta r^*$, $\int_\Gamma \epsilon = \int_{\bar{\Gamma}} \epsilon$. Hence, the following equivalences hold:

$$\begin{aligned}
\int_{\mathbb{R}^n} \bar{f}(x) &= \int_\Gamma \bar{f}(x) + \int_{\bar{\Gamma}} \bar{f}(x) + \int_{(\Gamma \cup \Gamma')^c} \bar{f}(x) \\
&= \int_\Gamma [f(x) - \epsilon] + \int_{\bar{\Gamma}} [f(x) + \epsilon] + \int_{(\Gamma \cup \Gamma')^c} f(x) \\
&= \int_{\mathbb{R}^n} f(x) - \int_\Gamma \epsilon + \int_{\bar{\Gamma}} \epsilon \\
&= \int_{\mathbb{R}^n} f(x) \\
&= 1.
\end{aligned}$$

□

Proof of Lemma 9. For any d and d' such that $d' > d$, we have that

$$\int_{\{M(r)-d'r\}^+} \bar{f}(x) = \int_{\{M(r)-d'r\}^+ \setminus \{M(r)-dr\}^+} \bar{f}(x) + \int_{\{M(r)-dr\}^+} \bar{f}(x).$$

By Lemma 8, $\bar{f}$ is strictly positive. Because the first term on the right hand side is the integral of a strictly positive function on a region of positive measure, it is strictly positive. Hence,

$$\int_{\{M(r)-d'r\}^+} \bar{f}(x) > \int_{\{M(r)-dr\}^+} \bar{f}(x).$$

$\square$

Proof of Lemma 10. If $\int_{M(r)^+} \bar{f}(x) < 1/2$, then $\bar{M}(r) = M(r) - dr$ for some $d > 0$ because the integral is strictly increasing in d by Lemma 9. Hence, $\bar{M}(r) \not\subset M(r)^+$.

If $\bar{M}(r) \not\subset M(r)^+$, then $\bar{M}(r) = M(r) - dr$ for some $d > 0$. By definition, $\int_{\{M(r)-dr\}^+} \bar{f}(x) = \frac{1}{2}$. Setting $d = 0$ implies $\int_{M(r)^+} \bar{f}(x) < \frac{1}{2} = \int_{M(r)^+} f(x)$ because $\int_{\{M(r)-dr\}^+} \bar{f}(x)$ is strictly increasing in d by Lemma 9. $\square$

Proof of Proposition 3. We show that for any direction r for which $r^* \cdot r \geq 0$, $\bar{M}(r) \subset M(r)^+$. First, by decomposing the region of integration, we see that

$$\int_{M(r)^+} \bar{f}(x) = \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\bar{\Gamma} \cap M(r)^+} \epsilon.$$

Consider r such that $r \cdot r^* \geq 0$ and choose any $x \in \Gamma \cap M(r)^+$. By definition, $x + \delta r^* \in \Gamma'$ and, further,

$$r \cdot (x + \delta r^*) = r \cdot x + \delta r \cdot r^* \geq r \cdot x \geq r \cdot m(r),$$

so $x + \delta r^* \in M(r)^+$. Hence,

$$x \in \Gamma \cap M(r)^+ \implies x + \delta r^* \in \bar{\Gamma} \cap M(r)^+.$$

Therefore, the measure of $\Gamma \cap M(r)^+$ is no larger than the measure of $\bar{\Gamma} \cap M(r)^+$. Thus, $\int_{\Gamma \cap M(r)^+} \epsilon \leq \int_{\bar{\Gamma} \cap M(r)^+} \epsilon$. Hence, we can now see that

$$\int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\bar{\Gamma} \cap M(r)^+} \epsilon \geq \int_{M(r)^+} f(x).$$

By Lemma 10, $\bar{M}(r) \subset M(r)^+$ and therefore $\bar{M}(r)$ is either equal to $M(r)$ or further from β.

We now show that the distance between median hyperplanes is bounded by δ if r is such that $r^* \cdot r < 0$. Consider the integral of the new density function on the positive halfspace of $M(r)$:

$$\int_{M(r)^+} \bar{f}(x) = \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\bar{\Gamma} \cap M(r)^+} \epsilon.$$

Consider $x \in \bar{\Gamma} \cap M(r)^+$. By definition, $x - \delta r^* \in \Gamma$ and, further,

$$r \cdot (x - \delta r^*) = r \cdot x - \delta r \cdot r^* > r \cdot x$$

because $r \cdot r^* < 0$. Because $r \cdot x \geq r \cdot m(r)$, it follows that $x - \delta r^* \in M(r)^+$. Thus, $x \in \bar{\Gamma} \cap M(r)^+ \implies x - \delta r^* \in \Gamma \cap M(r)$. Consequently, the measure of $\Gamma \cap M(r)^+$ is greater than or equal to the measure of $\bar{\Gamma} \cap M(r)^+$ and thus, $\int_{\Gamma \cap M(r)^+} \epsilon \geq \int_{\bar{\Gamma} \cap M(r)^+} \epsilon$.

If $\int_{\Gamma \cap M(r)^+} \epsilon = \int_{\bar{\Gamma} \cap M(r)^+} \epsilon$, then $\int_{M(r)^+} \bar{f}(x) = \int_{M(r)^+} f(x)$ and $M(r)$ is the median hyperplane when the distribution is $\bar{f}$. Hence, $d = 0$. If $\int_{\Gamma \cap M(r)^+} \epsilon > \int_{\bar{\Gamma} \cap M(r)^+} \epsilon$, then $\int_{M(r)^+} \bar{f}(x) < \frac{1}{2}$, and thus $\bar{M}(r) = M(r) - dr$ for some $d > 0$ by Lemma 10.

We now show that $d \leq \delta$.

Let $\mathbb{1}\{x \in \Gamma\}$ be an indicator function equal to one if and only if $x \in \Gamma$. Then

$$\int_{\{M(r)-dr\}^+} f(x) - \int_{\Gamma \cap \{M(r)-dr\}^+} \epsilon = \int_{\{M(r)-dr\}^+} f(x) - \epsilon \mathbb{1}\{x \in \Gamma\}.$$

The function $f(x) - \epsilon \mathbb{1}\{x \in \Gamma\}$ is strictly positive because $f(x) > \epsilon$. By a parallel argument to Lemma 9, $\int_{\{M(r)-dr\}^+} f(x) - \epsilon \mathbb{1}\{x \in \Gamma\}$ is strictly increasing in d. Therefore,

$$\int_{\{M(r)-\delta r\}^+} f(x) - \int_{\Gamma \cap \{M(r)-\delta r\}^+} \epsilon > \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon.$$

Next we show that $x \in \Gamma \cap M(r)^+ \implies x + dr^* \in \bar{\Gamma} \cap \{M(r) - \delta r\}^+$. By definition, $x + \delta r^* \in \Gamma$. What must be shown is that $x + \delta r^* \in \{M(r) - \delta r\}^+$. Observe that

$$r \cdot (x + \delta r^*) \geq r \cdot m(r) - \delta \iff r \cdot x + \delta r \cdot r^* \geq r \cdot m(r) - \delta.$$

The inequality on the right hand side of this equivalence holds because $r \cdot x \geq r \cdot m(r)$ and $r \cdot r^* \geq -1$. Thus, if $x \in \Gamma \cap M(r)^+$ then $x + dr^* \in \bar{\Gamma} \cap \{M(r) - \delta r\}^+$, and the measure of $\Gamma \cap M(r)^+$ is no greater than the measure of $\bar{\Gamma} \cap \{M(r) - \delta r\}^+$.

Now we consider the integral of over $\{M(r) - \delta r\}^+$:

$$\int_{\{M(r)-\delta r\}^+} \bar{f}(x) = \int_{\{M(r)-\delta r\}^+} f(x) - \int_{\Gamma \cap \{M(r)-\delta r\}^+} \epsilon + \int_{\bar{\Gamma} \cap \{M(r)-\delta r\}^+} \epsilon.$$

Because

$$\int_{\{M(r)-\delta r\}^+} f(x) - \int_{\Gamma \cap \{M(r)-\delta r\}^+} \epsilon > \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon,$$

we have

$$\int_{\{M(r)-\delta r\}^+} \bar{f}(x) > \int_{M(r)^+} f(x) - \int_{\Gamma \cap M(r)^+} \epsilon + \int_{\bar{\Gamma} \cap \{M(r)-\delta r\}^+} \epsilon.$$

Because the measure of $\Gamma \cap M(r)^+$ is no greater than the measure of $\bar{\Gamma} \cap \{M(r) - \delta r\}^+$,

$$\int_{\Gamma \cap M(r)^+} \epsilon \leq \int_{\bar{\Gamma} \cap \{M(r)-\delta r\}^+} \epsilon.$$

Thus,

$$\int_{\{M(r)-\delta r\}^+} \bar{f}(x) > \int_{M(r)^+} f(x) = \frac{1}{2}.$$

Because the left hand side of this inequality is strictly increasing in d, it follows that the d required to obtain a median hyperplane for the distribution $\bar{f}$ is no greater than δ. □

Proof of Theorem 5. (i) Because a is an RCW when the ideal points are distributed according to f, we know that $M(r) \subset H(a, b(r))^+$ for every $r \in \omega(a, \beta)$ by Proposition 2. By Proposition 3, when the ideal points are distributed according to $\bar{f}$, $M(r)' \subset M(r)^+$ and, hence, $M(r)' \subset H(a, b(r))^+$ for every $r \in \omega(a, \beta)$. Thus by Proposition 2, a is an RCW.

(ii) Because a is an SRCW when the ideal points hare distributed according to f, we know that $M(r) \subset H(a, b(r))^{++}$ for every $r \in \omega(a, \beta)$ by Proposition 2. It follows that $r \cdot m(r) > r \cdot \frac{a+b(r)}{2}$ for every $r \in \omega(a, \beta)$. By Proposition 3, $\bar{M}(r) = M(r) - dr$ for $d < \delta$. By choosing δ to be sufficiently small, $r \cdot \bar{M}(r) > r \cdot m(r) - \delta > r \cdot \frac{a+b(r)}{2}$ for every $r \in \omega(a, \beta)$. Therefore, $\bar{M}(r) \in H(a, b(r))^{++}$ for every $r \in \omega(a, \beta)$, so by Proposition 2, a is an SRCW when the ideal points are distributed with density $\bar{f}$. □

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